Basic Solar Positional Astronomy
Part 1: Essential Parameters and the Equation of Time

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The history of astronomy and timekeeping goes back many millennia. The terms used reflect this long history - and can be confusing to the non-astronomer. The author certainly became en-mired in this confusion – and this paper largely reflects how he sorted it out in his own mind. This paper hopes to chart some clarification.

The essential solar parameters needed by the gnomonist are:

- Right Ascension and Declination of the Sun – both Mean & True,
- Equation of Time, $^{\text{See Note 1}}$
- Altitude and Azimuth of the Sun,
- Time of Sunrise & Sunset.

Part 1 of the Series will define the basic astronomical terms that are needed, how Coordinated Universal Time & Greenwich Mean Sidereal Time are calculated and charts the route needed to calculate the Equation of Time.

Part 2 will detail a method that can be used to calculate the Right Ascension and Declination of the Sun and the other parameters above. The classical astronomical method based on Kepler's single body approach will be used. This approach is satisfying since, with only a few basic astronomical parameters, one may derive the parameters listed above with *far* greater accuracy than is generally required for the most sophisticated sundial design.

Part 3 presents a little Fourier theory and some simple formulae - derived by Fourier analysis - that allow rapid and accurate calculation of the Equation of Time, Declination and Right Ascension, for those who do not want to bother with the complete calculations

History

We will skip lightly over those thousands of years, when Unequal or Seasonal hours were in use. When Scientific or Common hours were introduced by the Arabs in late mediaeval times, time was told by the Solar Time, now called Local Apparent Time. Noon was when the sun was at its zenith. The vast majority of sundials still tell Common hours.

However, around the Enlightenment, with ever increasing international maritime trade, the navigators' need for accurate longitude determination spurred the need for clocks that ticked uniformly with the rotation of the Earth around the Equator. Such clocks tell Mean Time. However, the Sun moves around the Ecliptic at 23° to the equator and its elliptical orbit means that it does not appear to move uniformly. So there is an *imaginary* Sun - the Mean Sun,- moving uniformly around the Equator which takes the place of the real Sun and tells such time. One cannot see an imaginary sun. But, since the Stars do appear to move uniformly around the Equator, they are used to measure Sidereal Time. This, in turn, with a suitable conversion, is used to determine accurate Mean Time.

The discrepancy between Mean and Solar Time is called the Equation of Time. Ancient Greek astronomers understood this discrepancy and, around 150 AD, Claudius Ptolemy gave a succinct description of the geometries that give rise to this non-uniformity and methods with which to calculate it. It was not until the time of Kepler in 1621 that the Earth's elliptical orbit was fully understood and some years later, Newton showed that Kepler's theories could be explained by his Laws of Gravity.

Until the arrival of the telegraph - there was little option but to set one's clock by a sundial, albeit corrected, if needed, for the Equation of Time. It was not until the late 19th century, the introduction of the telegraph and the demands of the railway companies allowed cross-country dissemination of accurate mean time, determined by astronomers. Thus, bit-by-bit, Local Solar Time was gradually displaced by Local Mean Time and thereafter by National Mean Time. GMT was introduced in 1880 in the UK. The changes wrought by the subsequent conversion of GMT to Universal Coordinated Time (UTC) and the introduction of Atomic Time are of irrelevant magnitude to the gnomonist.

Background and Approach Taken

The elliptical nature of the solar orbit gives rise to one difference between Solar Time and Mean Time - which is approximately sinusoidal with a yearly period, phased with perihelion in January (when the Sun is closest to the Earth) and with magnitude of some 7.4 minutes. Calculating this difference is a problem of dynamics.

The 23.4° obliquity between the Ecliptic and the Equator gives rise to a second difference - which is somewhat sinusoidal with a six-monthly period, phased with the Vernal Equinox in March and with magnitude of some 9.9 minutes. Calculating this difference is a problem of spherical trigonometry.

The fact that most of us do not live on our Time Zone meridian (plus the introduction of Summer or Daylight Saving time) provides the third difference between Solar Time and that told by our watches. This correction involves a simple arithmetic calculation.

The calculation of the Sun's Altitude and Azimuth for any time/date and location is once again a problem of spherical trigonometry.

The traditional geocentric view is used - the Sun travelling around the Earth. While one 'knows' that the Earth revolves around the Sun, it is common to refer to the converse. It is only a matter of one's frame of reference. It makes no calculational difference when considering just the Sun & Earth. The Earth's longitude with respect to the Sun is just 180° difference from the Sun's longitude with respect to the Earth. On the other hand, a heliocentric view makes it much easier to explain the movement of the Planets in relation to the Earth.

Since this paper is meant to present the basics, it makes certain simplifications to definitions and equations consistent with the provision of results at levels of accuracy that are *more than sufficient* for the needs of the gnomonist. Pedants should read the notes at the end where I have tried to be more precise.

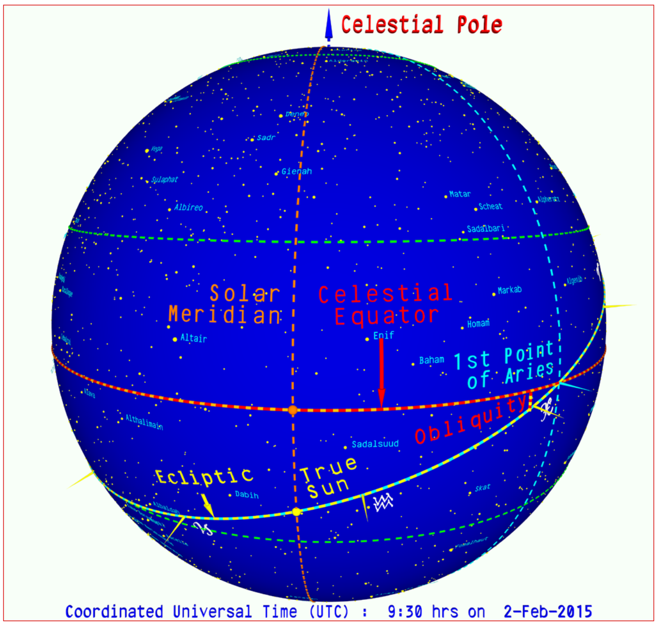


Fig. 1. The Celestial Sphere

Astronomical Nomenclature & Definitions

Since the Stars appear to rotate around the Earth with exemplary uniformity. See Note 2 24 hours of time equates to 360° of rotation. Hours and Degrees can be used interchangeably with a conversion factor of 15.

Traditionally, some parameters (e.g. Right Ascension) are quoted in hrs/mins/secs) and some parameters (e.g. Hour Angles) are quoted from -180° to +180°.

In all the figures and calculations below, parameters are in Degrees +ve West to East. This ensures a consistent arithmetic and the avoidance of sign errors. This is the international convention, though not always used in gnomonics, e.g. in the BSS Sundial Glossary Ref. 1. Otherwise, Glossary symbols are used throughout. A summary of the abbreviations and their translation is given in Table 1 towards the end of text. Definitions below are given on indented paragraphs.

The figures are correctly calculated for a given place, viz Athens - Time Zone 2 and for a given date/time - 2nd February 2013 at 11:30 a.m. local civil time. See Note 3

The Celestial Sphere

It has been practice throughout the ages to place the Earth at the centre of the Celestial Sphere. Fig. 1 shows the Celestial Sphere viewed from the medieval Empyrean - the place outside the Stars - where God is.

> The Celestial Sphere is an imaginary sphere of arbitrarily large radius, concentric with the Earth and rotating upon the same axis. All objects in the sky can be thought of as projected upon the celestial sphere. The celestial equator and the celestial poles are the outward projections of the Earth's equator and poles.

The Ecliptic at 23.4° from the Celestial equator is the path around which the Sun appears to move.

An essential point on the Celestial Sphere is one of the two intersections of the Celestial Equator and the Ecliptic. The point chosen is the point when the Sun crosses the celestial equator during the northern hemisphere spring and is called the Vernal Equinox. Somewhat confusingly, it is also called the First Point of Aries. These terms are used more-or-less interchangeably. Strictly speaking, the First Point of Aries is a direction in the sky, while the Vernal Equinox is a moment of time. The First Point of Aries is the prime origin for all measurements made along the Celestial Equator and the Ecliptic. Confusingly, the First Point of Aries is no longer in the astronomical Constellation of Aries. It was - in classical Greek times - but as a result of Precession see Note 4, it is now in the Constellation of Pisces. See Note 5

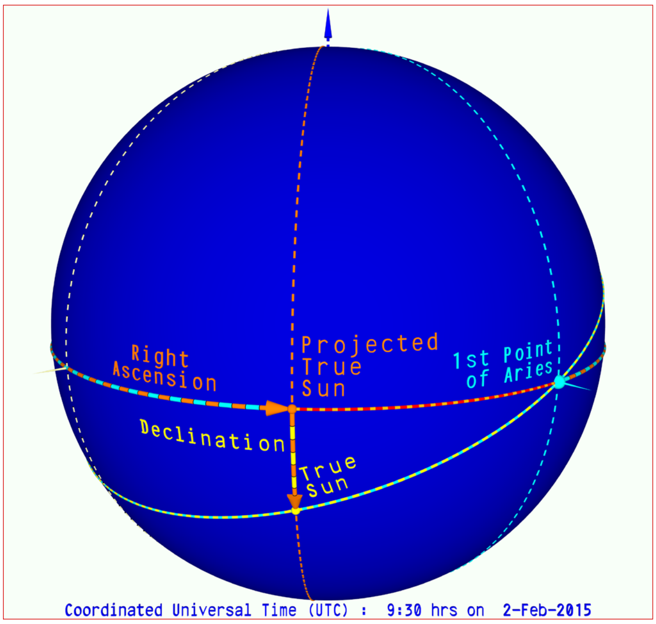


Fig. 2. Declination & Right Ascension

Zenith & Meridian

The Zenith is the point on the Celestial Sphere directly above the observer. (The opposite point on the Sphere is the Nadir).

A meridian is a great circle on the celestial sphere that passes through the North & South Celestial Poles and either through a point on the Celestial Sphere or through the Zenith of an observer on the Earth's surface.

Meridians are analogous to line of longitude on the Earth's surface. Angles between meridians (as angles between lines of longitude) are measured around the Celestial Equator.

Right Ascension & Declination

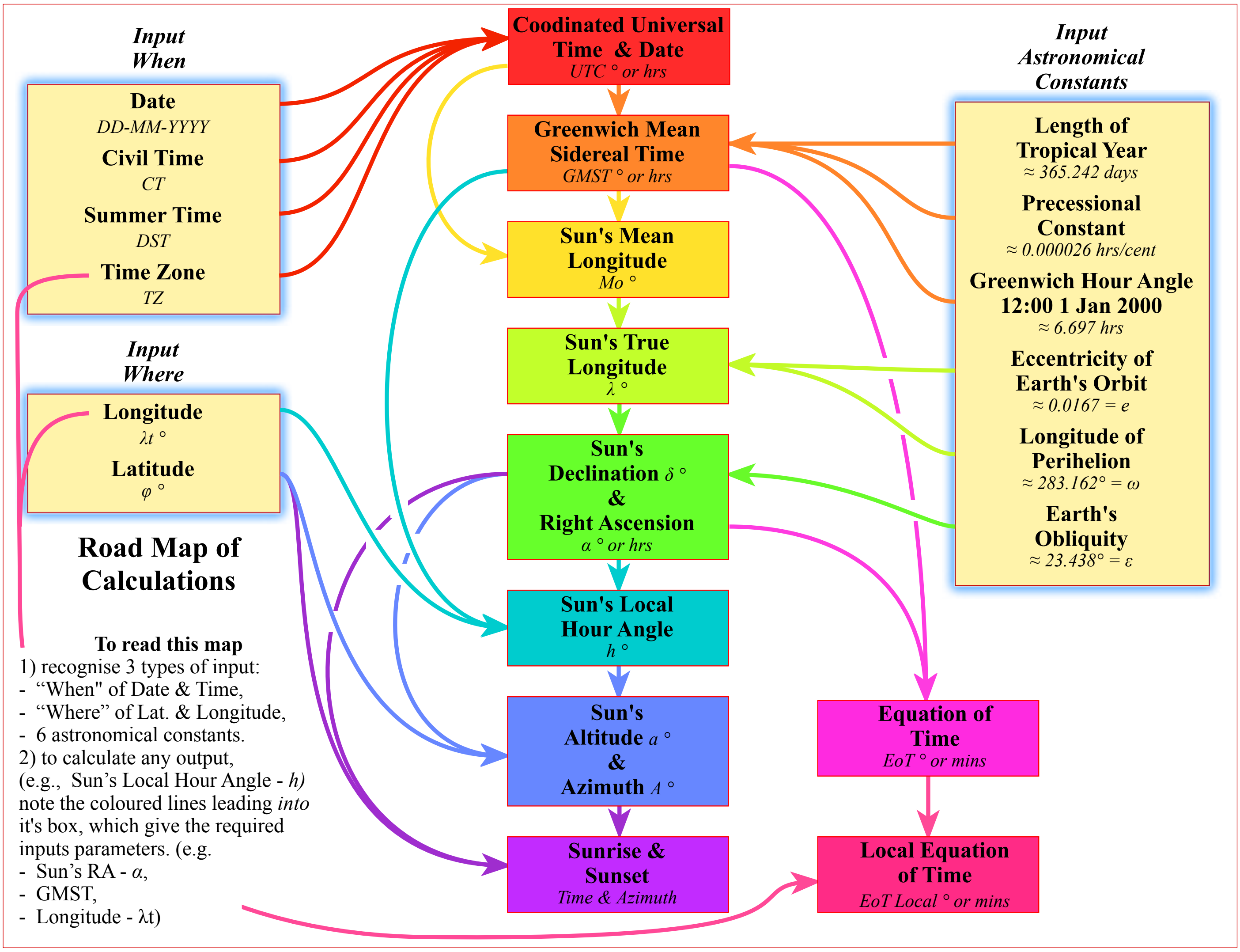
We are concerned with the position of the Sun on the Celestial Sphere. This is measured by Right Ascension & Declination. See Fig. 2. These are equivalent to our terrestrial Longitude & Latitude, except that...

- Declination uses the Celestial equator, running from +90° to -90° - positive towards the north, negative towards the South.
- Right Ascension is measured along the celestial equator and the 1st Point of Aries as origin. It is measured anti-clockwise - when viewed from the North Celestial Pole. This is the direction in which the Earth rotates and in which the Sun appears to move. Traditionally, RA is quoted in Hours/Minutes/Seconds, running from 0 to 24 hrs. But Degrees are generally used in this paper.

The Sun moves around the Ecliptic at very approximately 365/360° per day, so its RA and Decl are continuously changing. In Part 2 of this series, we will see how solar dynamics can used to calculate the Sun's RA & Declination for any given time and date.

In passing, we should note that...

- the planets (from Greek πλανήτης αστήρ "wandering star") move near to the Ecliptic in somewhat erratic manner (from a geocentric point of view) so their RA & Decl are also continuously changing.
- the RA and Decl of any star is effectively constant. See Note 6
- RA and Decl have nothing to do with the daily spinning of the Earth about its axis.



To read this map

1) recognise 3 types of input:
- "When" of Date & Time,
- "Where" of Lat. & Longitude,
- 6 astronomical constants.

2) to calculate any output, (e.g., Sun's Local Hour Angle - *h*) note the coloured lines leading *into* it's box, which give the required inputs parameters. (e.g.
- Sun's RA - *α*,
- GMST,
- Longitude - λt)

Fig. 3. The Calculation's Road Map

What are we trying to Calculate...

Fig. 3 charts the path along which calculations are made. The start is made by provision of three classes of input…

- the "When", the local time and date;
- the "Where", the terrestrial Latitude and Longitude of the Observer (or the Sundial);
- the 6 astronomical constants required – 3 of which are not quite constant.

In this part of the series,

- the simple connection between local Civil Time and date – which we hear on the radio and read from our watches – and Coordinated Universal Time – UTC – is established.
- the more complicated connection between Greenwich Mean Sidereal Time - GMST - and UTC is made
- the connection between UTC and Sun's Mean Longitude is made
- the formulae to establish the Equation of Time and the Longitude Correction is introduced.

Coordinated Universal, Standard & Civil Time

Some gnomonists eschew civil time and rely entirely on 'true' or Solar time – it is noon when the Sun is South. The author respects this view. But he personally feels it is of paramount importance that the gnomonist should be capable to explain to our young why the sundial reads a different time to that on their watch or mobile phone. Hence the apparently perverse starting point of Civil – rather than Solar - Time.

It was the advent of the railways that forced society to adopt mean time so that the same time was used everywhere in a country (or in large portions of a country – as in Russia or the USA). The global starting point was Greenwich Mean Time – GMT. This has morphed, with minor changes, in Coordinated Universal Time – UTC.

> Coordinated Universal Time (UTC) is 12 + the hour angle at Greenwich of the Mean Sun. The hour angle being converted from degrees to hours at 360°/day.

Although the 'tick' of UTC now relies on atomic clocks, its formal definition is in terms of the Mean Sun. The Mean Sun – which is an imaginary body…

> The Mean Sun is an abstract fiducial point at nearly the same Hour Angle as the Sun, but located on the mean celestial equator of date and characterized by a uniform sidereal motion along the equator at a rate virtually equal to the mean rate of annual motion of the Sun along the ecliptic.

As an example, when the Mean Sun's meridian has moved west by 15° (or 1 hour) from the Greenwich meridian, UTC = 12 + 1 = 13:00 ^{hrs}, which is what one would expect. The term 'fiducial' is the technical term for a point that is a fixed and trusted basis for comparison. In simple terms...

- the *mean* sun is an *imaginary* body that *uniformly* moves around the *Equator,* once in one tropical year.

on the other hand...

- the *true* Sun, moves *non-uniformly* around the *Ecliptic*, once in one tropical year. The true sun is thus 'out-of-angle' with the axis which creates our day/night.

In passing, we should note that...

- the 'Tropical' Year is the time taken for the sun *(on average)* to pass through the 1st Point of Aries - 365.242 191 days. Note that our leap year system gives a 'Calendrical' Year of (365.25 x 400 - 3) / 400 = 365.242 500 days, which closely matches the length of the Tropical Year, ensuring that the Calendar does not drift away from the Seasons.
- Atomic Time is kept in sync with the solar definition of UTC by the occasional insertion of Leap Seconds, which compensate for the gradual slowing of the Earth's rotation.

UTC is a surrogate for Solar time in providing a universal and uniform time scale. The Mean Sun's position has zero declination and its Right Ascension increases uniformly from 0° at the Vernal equinox to 360° at the next Vernal equinox.

In the 1880s, Greenwich Mean Time was established as legal time across the UK. Other countries offset their own mean time by integral number hours (or half hours) before or after Greenwich - thus introducing the Time Zones. So Standard Time was created. Greenwich Mean time morphed with minor changes into Coordinated Universal Time (now UTC).

> Standard Time – ST - is Mean Time on the Time Zone meridian of that area. Time Zone meridians are (usually) in 15° Longitude increments away from the Greenwich meridian.

Standard Time may be further moderated by the introduction of Summer or Daylight Saving to give Civil Time - CT. In winter, Civil Time is the same as Standard Time. Civil Time is the legal binding time in a given Time Zone.

$$UTC^{hrs} = ST^{hrs} - Time\ Zone^{hrs\ (+ve\ East\ of\ Greenwich)} \qquad \text{........... Equ 1.1}$$

$$UTC^{hrs} = CT^{hrs} - Time\ Zone^{hrs} - DST^{hrs} \qquad \text{................. Equ 1.2}$$

Calculations of solar positions need both a time and a date, and it must be recognised that if the correction in Eqn. 1 lead to a different day in Greenwich than that of the observer, a correction is needed...

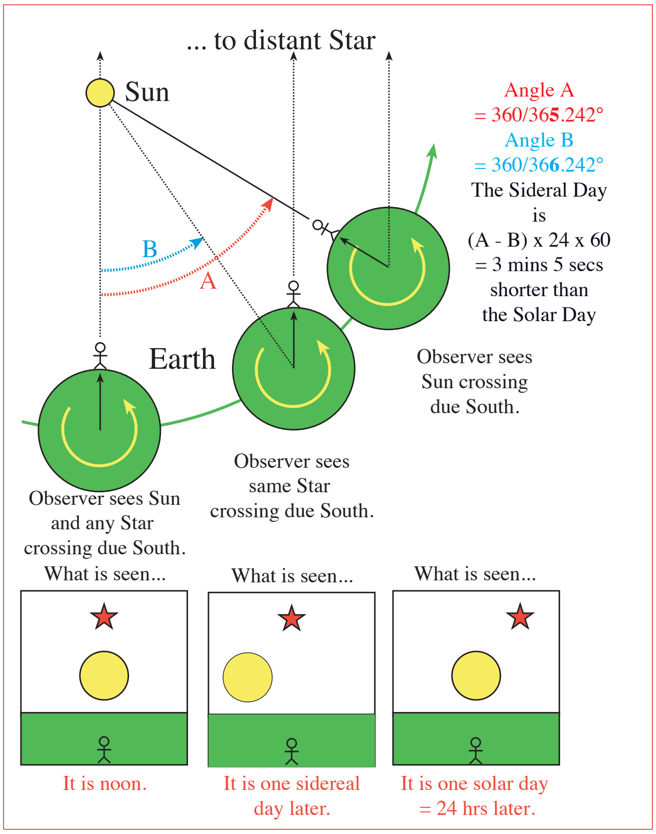


Fig. 4. Sidereal Time -v- Solar Time

$$\text{if } UTC^{hrs} > 24$$
$$UTC^{hrs} = UTC^{hrs} - 24 \ \& \ Date^{day} = Date^{day} + 1$$
$$\text{if } UTC^{hrs} < 0$$
$$UTC^{hrs} = UTC^{hrs} + 24 \ \& \ Date^{day} = Date^{day} - 1$$

................Eqns 1.3

Finding Greenwich Mean Sidereal Time

Before atomic clocks, the problem with GMT was that it was based on an imaginary mean Sun. Thus it was not measurable, especially by navigators trying to calculate longitude. They require an entirely uniform, definable and measurable time scale that accords with the axis of spin of the Earth and which is independent of the vagaries of the Sun's apparent movement. This is provided by the stars - so-called Sidereal Time (from the Latin word 'sidus' meaning 'star').

On successive nights, it is easy to measure 'transits' of any star i.e. when it has its highest altitude in the sky. Thus the stars began to be used as time-keepers and so-called sidereal day was defined by successive transits of any star through an observer's meridian. The introduction of Sidereal time was the start of the gradual decline of Sundials as civilization's primary time keeper. Astronomers – rather than gnomonists – gradually became Masters of Time

The sidereal day is not the same as the solar day. Fig. 4 shows a solar day, *defined by the transit of the sun*, as compared with the sidereal the day, *defined by the transit of a star*. There are 366.242 transits of a given star in the same time as 365.242 transits of the sun. This is because the Sun itself has circled one revolution against the stars. The ratio 366.242/ 365.242 = 1.002738 will crop up again in our calculations.

Against this background,

> Greenwich Mean Sidereal Time (GMST) is the angle along the celestial equator from the Mean Vernal Equinox (1st Point of Aries) to the Greenwich meridian.

Both Sidereal Time and UTC record an evenly ticking cycle that completes each tropical year. Therefore, it is possible to define UTC explicitly in terms of Sidereal Time. This definition is 'owned' by the International Astronomical Union.

$$\begin{aligned} GMST^{hrs} = (&6.697\,374\,558^{hrs} \\ &+ 0.065\,709\,824\,419\,08 \times D_0^{days} \\ &+ 1.002\,737\,909\,35 \ \times UTC^{hrs} \\ &+ 0.000\,026 \times T^2) \bmod 24 \qquad \text{............. Eqn 1.4} \end{aligned}$$

D_0 is the number of days from $12{:}00^{hrs}$ on 1st January 2000 – the so-called Epoch$_{2000}$.- until the mid-night that starts the day in question. T is the number of Julian Centuries of 36,525 days from Epoch$_{2000}$ until the moment of time in question. The 'mod' function reduces the answer to fall between 0 and 24 hours. This is a slight simplification of the complete definition. For ultimate but unnecessary accuracy... [See Note 10].

The numbers in this definition are not arbitrary.

- 6.697 374 558 was the Greenwich hour angle of the Sun at Epoch$_{2000}$.
- 0.065 709 824 419 08 $= 24^{hrs/day} \,/\, 365.242\,191^{days/tropical\ year}$

  which ensures that, in one tropical year, GMST increases by 24 hours, corresponding to the extra sidereal day in the tropical year.

  D_0 is the number of days from Epoch$_{2000}$ to midnight of the day in question.
- 1.002 737 909 35 $=366.242\,19^{sidereal\ days/year}/365.242\,191^{tropical\ days/year}$

  this converts from normal to sidereal hours.
- 0.000 026 x T^2 accounts for Precession. [See Note 4]

  T is the number of Julian Centuries (of 36525^{days}) from the Epoch$_{2000}$.

  Note that three of the six input astronomical constants are involved in this definition.

Since our years and months are of variable length, any given date and time combination is not directly amenable to mathematical formulae, so a strictly linear time/date scale is used throughout the astronomical world. This is the Julian Date (JD).

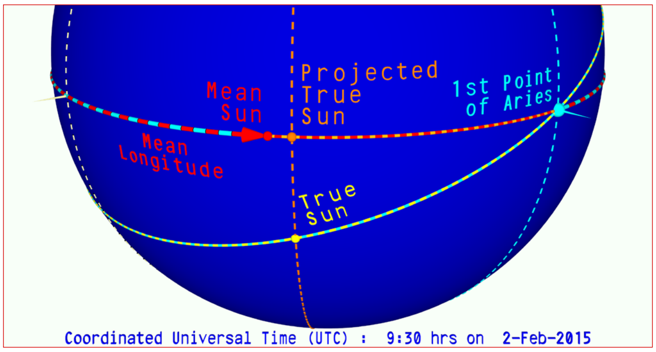


Fig. 5. The Mean Sun & Mean Longitude

The Julian Date is the number of decimal days that have elapsed since noon coordinated universal time (UTC), 1st January, 4713 BC. See Note 8. However for these calculations, times from Epoch$_{2000}$ (12:00 hrs UTC on 1st January 2000) are needed, which is the Julian Date reduced by 2451545.0

In passing, we may note that...

$$Date^{days}_{Epoch\ 2000} = JD^{days} - 245\,154\,5.0^{days} \text{ .... Eqn 1.5}$$

Date/Time$_{\text{Greenwich}}$ is given by YYYY$^{\text{years}}$, MM$^{\text{months}}$, DD$^{\text{days}}$, HH$^{\text{hrs}}$, MM$^{\text{mins}}$ then to obtain the D_0 - *during this century* - apply the following formula:

$$\begin{aligned}
bbb &= 367 \times YYYY - 730\,531.5 \\
ccc &= -\operatorname{int}\Big(\big(7 \times \operatorname{int}\big(YYYY + (MM + 9)/12\big)\big)/\ 4\Big) \\
ddd &= \operatorname{int}\big(275 \times MM / 9\big) + DD \\
D_{today} &= \big(HH + MM / 60\big)\ /\ 24 \\
D_0 &= a + b + c^{\text{See Note 9}} \\
T &= (D_0 + D_{today})\ /\ 365\,25 \qquad \text{.................. Eqn. 1.6}
\end{aligned}$$

Why these formulae work is a mystery to the author... The ‘int’ function removes the fractional part of the calculation just made. The ‘mod’ function reduces the result until it lies between 0 & 24.

Finding the Sun’s Mean Longitude

Referring once more to Fig. 3, the next thing to calculate is the Mean Sun’s Longitude. This may also be referred to as the Mean Suns; Right Ascension. It is measured along the Celestial equator, from the 1st Point of Aires See Figs. 5 & 6.

In the latter, working from out to in, see the various arcs...

- the Sun’s Mean Longitude - M_O - origin 1st Point of Aries
- GMST - origin 1st Point of Aries
- UTC - origin at the Nadir (the opposite point) from the Mean Sun. This reflects the definition of UTC (see above) - or more obviously the fact that our 0:00$^{\text{hrs}}$ at midnight is 180° away from mean noon, the moment when the Mean Sun’s Hour Angle is 0°
- complimentary arc 180 - UTC

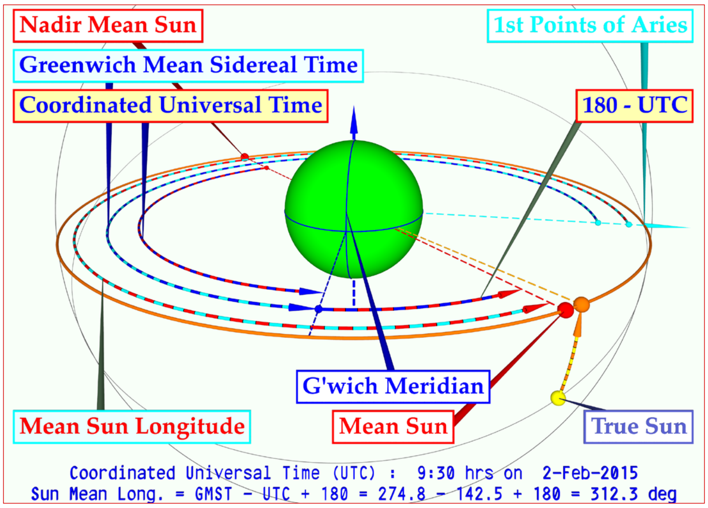


Fig. 6. GMST, UTC & Mean Longitude

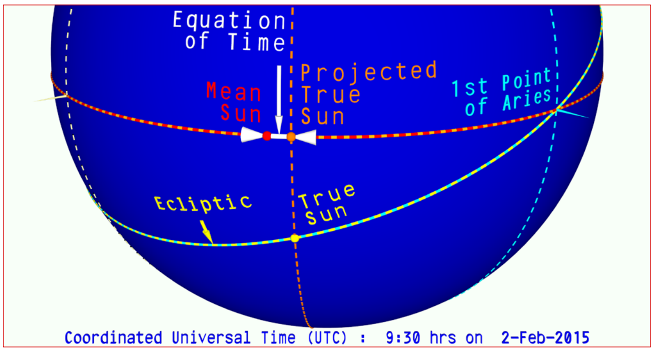


Fig. 7. The Equation-of-Time

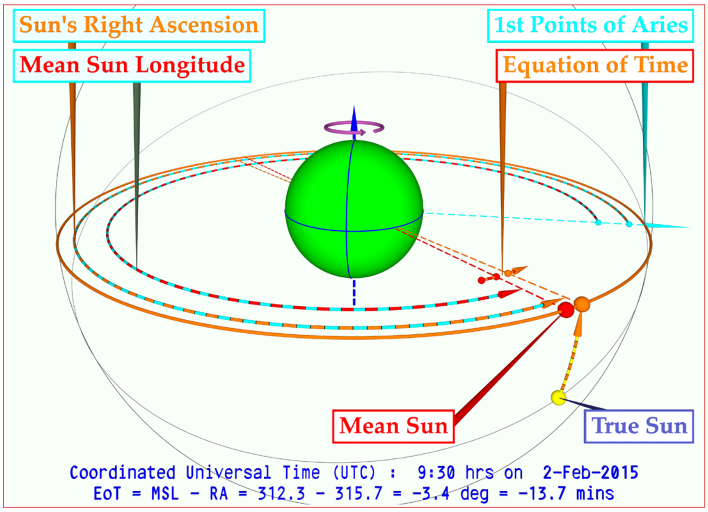


Fig. 8. The Equation of Time - see Eqn. 1.8

From the figure, it is apparent that...

$$M_0^{deg} = GMST^{deg} - UTC^{deg} + 180^{deg} \text{ .... Eqn. 1.7}$$

Introducing the Sun’s Right Ascension and the Equation of Time

The Sun’s Right Ascension was introduced above, see Fig. 2. Putting this together with the definition of Mean Longitude, we can find the Equation of Time. See Figs. 7 & 8. From the arcs in Fig. 7, it may be seen the Equation-of-Time

$$EoT^{deg} = M_0^{deg} - \alpha^{deg} \text{ .... Eqn. 1.8}$$

Combining Eqns. 1.7 & 1.8...

$$EoT^{deg}_{astronomical} = GMST^{\,deg} - \alpha^{deg} - UTC^{\,deg} + 180^{deg} \text{... Eqn. 1.9}$$

All of these are explicitly known except for the Right Ascension of the Sun. This will be computed in Part 2 of this series. Those interested in gnomonics tend to use the inverse of this definition (i.e. the correction to be made to sundial time to get mean time) and want the results in minutes, thus...

$$EoT^{\,mins}_{gnomonical} \;=\; -\;4 \text{ x } EoT^{\,deg}_{astronomical} \text{ ....Eqn. 1.10}$$

In passing we may note that...

the formal definition from the all powerful Explanatory Supplement to the Astronomical Ephemeris and the American Ephemeris & Nautical Almanac $^{\text{Ref. 2}}$, is:

> .. As from 1965….. The equation of time will then be defined as the correction to be applied to 12h + Universal Time to obtain the Greenwich Hour Angle Sun,......... ; it is now so tabulated in the almanacs for navigators and surveyors…

This implies...

$$12^{degs} + UTC^{degs} + EoT^{degs}_{astronomical} \;=\; GHA^{degs}_{Sun} \text{ or}$$

$$EoT^{degs}_{astronomical} \;=\; GHA^{degs}_{Sun} - UTC^{degs} - 12^{degs} \text{ .....Eqn. 1.11}$$

But, by definition...

$$GMST \qquad\qquad = \text{angle } 1^{st}\text{Pt. of Aries} \Rightarrow \text{G'wich Meridian}$$

$$\alpha_{Sun} \qquad\qquad = \text{angle } 1^{st}\text{Pt. of Aries} \Rightarrow \text{Sun}$$

$$\therefore GMST - \alpha_{Sun} = \text{angle G'wich Meridian} \Rightarrow \text{Sun} = GHA_{Sun}$$

..........Equ1.12

Thus Equation 1.19 is the same as Eqn. 1.12

The Longitude Correction

Solar noon at 1° west of a Time Zone meridian is 4 mins of time after Solar noon on the Time Zone meridian. Thus, if we wish to correct our sundials to provide what our watches read, we must apply an additional offset - the Longitude Correction ...

$$\sigma^{deg} \;=\; Time\ Zone^{hrs} \times 15^{deg/hr} \;-\; \lambda^{deg}_{t} \text{ ....Eqn. 1.13}$$

So we may conclude that - if we coin a new term...

$$EoT^{mins}_{Local} \;=\; EoT^{mins}_{Gnomonical} + \sigma^{mins} \text{ ....Eqn. 1.14}$$

For a standard sundial (i.e. one whose hour lines are not longitude corrected and whose noon line on the North/ South meridian), it is suggested that any correction tables or graphs should indicate EoTLocal, with the additional comment that DST Hours should be added in the Summer.

Summing up

Table 1, below, sums up the various formulae, presented above. It can be seen that, for at any date/time/location, all the parameters can be deduced or calculated from one another - provided that the Right Ascension of the Sun can be found. These calculations, together the conversion to Azimuth and Altitude, Sunrise and Sunset. will be presented in Part 2 of this series.

Parameter	Symbol	Formula in degrees	Example	
Date		given	2nd Feb 2013	
Observer’s Longitude, +ve east of Greenwich	LON or λ_t	given	23.717°	23° 43’ 00”
Observer’s Time Zone, +ve east of Greenwich	TZ	given	60°	2 ^{hrs}
Observer’s Summer Time or Daylight Saving Hours	DST	given	0°	0 ^{hrs}
Observer’s Civil Time	CT	given	172.500°	11:30 am
Observer’s Standard Time	ST	CT - DST	172.500°	11:30 am
Coordinated Universal Time	UTC	ST - TZ	142.500°	9:30 am
Greenwich Mean Sidereal Time Calculated in terms of Date & UTC	GMST	(see Eqn. Set 1 & Eqn. 2)	274.761°	18^{hr} 19^{min} 02^{sec}
Sun’s Right Ascension Calculated in terms of Date & UTC	RA or α	(see Part 2)	315.673°	21^{hr} 02^{min} 41^{sec}
Equation of Time: Local Mean to Dial Time (Astronomical Convention) See Note below	$EoT_{Astronomical}$	GMST − α − UTC +180° = GMST − α − (CT - DST - TZ) + 180°	-3.413°	-13 ^{min} 39 ^{sec}
Equation of Time: Dial to Local Mean Time (Gnomonist’s Convention)	$EoT_{Gnomonical}$	- $EoT_{Astronomical}$	3.413°	13 ^{min} 42 ^{sec}
Longitude Correction	σ	TZ - λ_t	6.283°	25 ^{min} 08 ^{sec}
Equation of Time: Dial to Standard Time	EoT_{Local}	$EoT_{Gnomonical}$ + σ	9.696°	38 ^{min} 47 ^{sec}

Note: The Equation of Time calculated in this way may - depending on the time of day and year - give spurious looking results as a result of the cross-over from 24 hrs back to 0 hours. To correct, if EoT ^{mins} < -36 then add 48, if EoT ^{mins} < -12 then add 24.

Table 1. Basic Calculations

Notes

1. Various astronomical terms use the qualifier 'equation of…': the equation of time, the equation of centre, the equation of the equinoxes, the equation of origins, the equation of light. The term coming from Greek to Arabic to the mediaeval Latin 'equato' as in Equato Diem for EoT. In all cases, 'equation of…' means the difference between what is observed and the mean values of the phenomenon in question.
2. The Earth's rotation is not completely uniform. Not only does the position of the North and South Poles wander, but the rate of rotation is slowing in a somewhat random fashion by a number of seconds per decade. This is believed to be caused by tidal friction and crustal movements. This gives rise to the inclusion of 'leap seconds', mentioned in Note 7.
3. Two free software packages : 'Persistence of Vision', a precise 3-D simulation package & a precise 2-D NodeBox were used to prepare the graphics. The data required to draw the Stars in Fig. 1 was derived from the Right Ascension & Declinations of the 1000 brightest stars, readily found on the internet. All the figs used precisely drawn in accordance to the routines described in this document & Part 2 of the series.
4. Nothing on Earth or the Heavens is moving uniformly... In particular, the Earth's axis is slowing gyrating like an out-of balance spinning top. This effect - called Precession - has a long period of 25,600 years. It is caused by the torque induced by the Sun & Moon's gravitational pull on the equatorial bugle in the Earth's shape. Over time. Precession moves the position of the Vernal Equinox through the Sky. Most of the significant effect of precession, in these calculations is subsumed in the definitional formula for Mean Time. In addition to Precession - and primarily because of tidal forces between the earth and the moon - the axis of the earth is vibrating such there are complex minor variations in the position of the Vernal Equinox and the Obliquity of the axis. This is called Nutation. The effects are minor in the context of this paper. But precession and nutation lead to some potentially confusing nomenclature within astronomy. The terms **mean** equator, **mean** obliquity, **mean** equinox, **mean** sidereal time indicate that the effects of nutation are averaged out. (However, **mean time** has an entirely different context.) The term...**of date** indicates that precession has been considered, while...**of Epoch** refers to mean values on 1 January 2000, thus without precession. The term **apparent** indicates that all precessional, nutational and any other effects have been taken into account - i.e. it is what you will actually get on a given date/time.
5. The Reader should not confuse the astronomical *Constellation* of (e.g.) Pisces with the astrological *House* of Pisces. The two were the same in antiquity. The astrological Houses split the year into 12 equal portions starting at Aries on the Vernal Equinox. This is tropical astrology. However there is another branch - called Sidereal astrology, which does recognise the shift in constellations due to Precession.
6. In fact, since our galaxy is expanding, the stars do move relative to one another - their so-called 'proper motion' - but at usually imperceptible rates, unless they are close to the Sun. For example, the declination of our second closest star Alpha Centauri is changing at some 13 seconds of arc per year
7. The current basis for international timekeeping is Temps Atomic International (TAI). This is kept by an array of some 200 atomic clocks, kept in 30 countries around the world. These clocks 'tick' using the vibrations of the Cesium atom. The international standard second is the time taken for 9,192,631,770 cycles of radiation emitted during the transition between two hyperfine levels of the ground state of cesium 133 at 0° Kelvin. 24 x 60 x 60 x 365.242198781 of these original atomic seconds were matched to the length of the tropical year in 1900.

   The practically used time standard is Coordinated Universal Time (UTC) = TAI + a number of 'leap seconds', which are added to correct for the slight slowing of the Earth's rotation. This correction is made to maintain the historic and cultural/religious connection needed to align timekeeping with the 'tick' of the average solar day There have been 35 leap seconds added since 1971. As far as the gnomonist is concerned, UTC equates to the old Greenwich Mean Time - a term now abandoned.

   In order to sense when leap seconds are required and for other astronomical reasons, a further time scale confusingly called Universal Time (UT) is counted from 0 hours at midnight, with the unit of duration of the mean solar day. This is measured by observing the daily motion or various starts and extraterrestrial radio sources. The measured time is called UT0, which is then corrected to UT1, to account for the wobbling of the earth as a result of polar motion. The difference between UT1 (the 'astronomical' tick and UTC (the 'atomic' tick) is referred to as Delta T. Daily values of Delta T are published every week and forward forecast for 6 months. If Delta T exceeds 0.8 seconds, a further leap second will be introduced either on the following 30 June or 31 December.

   Moves - mostly from the computing industry - to abandon Leap seconds have led to an international symposium in 2012. Decisions have been deferred. China consider it important to maintain a link between civil and astronomical time due to Chinese tradition. This may be the clinching argument.

   The serious student of time or of planetary movement must also know all about Terrestrial Time (TT), Geocentric Coordinate Time (TCG), Barycentric Dynamical Time (TDB) and Barycentric Coordinate Time (TCB). These are generally concerned with the relativistic components of time keeping.
8. The Julian date system was invented by Joseph Justus Scaliger (1540-1609), a French classical scholar, in 1582, when he invented the Julian period, named after his father, Julius Caesar Scaliger. This was a period of 7,980 = 28 x 19 x 15 years.
   - 28 is the number of years in the Julian calendar that it takes for dates to fall again on the same days of the week, the so-called Solar cycle.
   - 19 is the number of years in the Metonic cycle, devised by Meton of Athens in 432 BCE, although known in China as early as 2260 BCE. The basis of ancient Greek, Jewish, and other calendars, it shows the relationship between the lunar and solar year. In 19 years of exactly 365.25 days each (the Julian, or solar year), there are 235 lunar cycles, with seven of these years having a 13th, or embolistic, month. At the end of the cycle, the phases of the moon recur on a particular day in the solar year. The Metonic cycle was important because it established a lunar calendar having a definite rule for intercalary months, and didn't get out of phase with the cycle of tropical (seasonal) years.
   - 15 is the number of years in the ancient Roman cycle of Indiction, a 15-year period used for taxation. It was used by Emperor Constantine beginning in 312 CE, and continued not only during the Middle Ages, but was used in the Holy Roman Empire until Napoleon abolished it in 1806.

   Scaliger chose 12:00 UT, 1 January 4713 BCE as the day 0.0 of the Julian system, since it was the nearest past year when all three cycles - Solar, Metonic and Indiction - exactly coincided. The present Julian period will end at 12:00 UT, 31 December 3267. (Adapted from Ref. 7.)
9. The observant reader will note that the introduction of Julian Date is not strictly necessary. It has been included since it is a frequently used astronomical term. In this case, the numbers a,b,c, & d are all that are required - providing the days since 1st Jan 2000.

10. This equation is an approximation - but good to 0.1 secs over the current century, see Ref. 8. For the ultimate precision, see Ref. 9 and the IAU SOFA computational routines in Ref. 10.
11. For greater precision, one may follow the route taken by the US Naval Observatory's MICA [Ref. 14] program uses the expression...

$$EoT^{hrs}_{Astronomical} = GMST^{hrs} + EoE^{hrs} - 12^{hrs} - UTC^{hrs} - RA^{hrs}_{Sun}$$

.... Eqn. 1.15

RA is Apparent Geocentric, True Equator and Equinox of Date. See Note 4 for meaning of apparent and of Date.

EoE is the Equation of Equinox, which is a small correction to account for nutation (typically of +/- a few seconds).

12. "Now let me see," the Golux said, "if you can touch the clocks and never start them, then you can start the clocks and never touch them. That's logic, as I know and use it..." James Thurber in *The 13 Clocks*.

References

There are a vast number of useful books on Astronomy....

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2. *Explanatory Supplement to the Astronomical Ephemeris & American Ephemeris and Nautical Almanac:* Her Majesty's Stationery Office (1961)
3. *The Astronomical Almanac for the Year 2010:* London, The Stationary Office (2009)
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5. G.J. Toomer: *Ptolemy's Almagest:* Princeton Press (1998).
6. R.M. Green: *Spherical Astronomy:* Cambridge University Press (1985)
7. Tom Alburger: *The Origin of Julian Days* : www.magma.ca/~scarlisl/DRACO/julian_d.html
8. *Approximate Sidereal Time* : US Naval Oceanography Portal: http://www.usno.navy.mil/USNO/astronomical-applications/astronomical-information-center/approx-sider-time
9. George H. Kaplan :*The IAU Resolutions on Astronomical Reference Systems, Time Scales & Earth Rotation Models* : US Naval Observatory, Circular 179
10. *Standards of Fundamental Astronomy* : IAU SOFA : http://www.iausofa.org

I used the following to calibrate and verify my calculations...

11. Peter Duffet-Smith: *Practical Astronomy with your Calculator:* Cambridge Univ. Press, Cambridge (1988).
12. Jean Meeus: *Astronomical Algorithms:* Willman-Bell, Richmond (1998).
13. *Horizons Software:* NASA/JPL: (2012) http://ssd.jpl.nasa.gov/?horizons

    This software uses JPL's DE405 routines which are the gold standard for Solar & Planetary ephemerides.
14. *Multiyear Interactive Computer Almanac - 1800 - 2050:* US Naval Observatory: (2012). This is a high precision astronomical program, that (e.g.) provides EoT to an accuracy of 0.1 second.

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September 2014

Basic Solar Positional Astronomy
Part 2: Calculating the Sun's Right Ascension, Declination & EoT

KEVIN KARNEY

Calculations Required

In Part 1 of this Series, we learnt how to calculate the Greenwich Mean Sidereal Time - GMST, together with the formulae needed to calculate the Equation-of-Time - EoT. In this part we will see how the Sun's actual position in the sky may be found, in terms of...

- the Ecliptic: the Sun's Longitude - λ
- the Equator: its Right Ascension - α or RA - and Declination - δ
- the Local Hour Angle - h
- the Horizon: its Altitude -a - and Azimuth - A
- the approximate times of Sunrise - h_{sr} and Sunset - h_{ss}

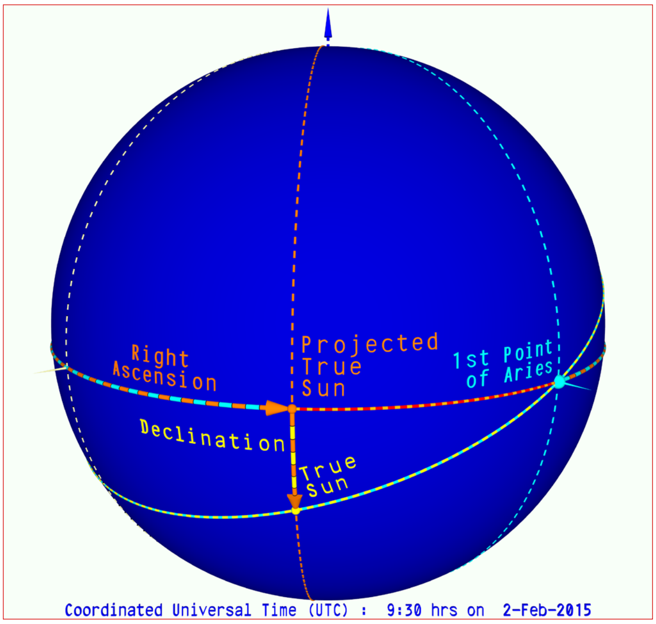

Fig. 1 It is required to find the actual position of the Sun - in terms of Declination & Right Ascension. The True Sun projected onto the Celestial equator provides the Right Ascension.

Once the RA is found, the Equation of Time can be computed.

$$EoT^{deg}_{astronomical} = GMST^{deg} - \alpha^{deg} - UTC^{deg} + 180^{deg}$$ ....Eqn. 2.1

Figs 1 to 3, repeated from Part 1, show illustrates the essential definitions and show graphically the Equation of Time.

There are two steps in calculating the Sun's Right Ascension & Declination, it is necessary to...

(i) find its position on the Ecliptic. This is the Solar Longitude - λ - which is measured around the Ecliptic, with 0° at the 1st Point of Aries. This is a dynamical problem.

(ii) convert the Solar Longitude (measured around the Ecliptic) to Declination - δ - and Right Ascension - α - (measured around the Equator, but also with 0° at the 1st Point of Aries.)

Figs 4 to 8 show these steps graphically.

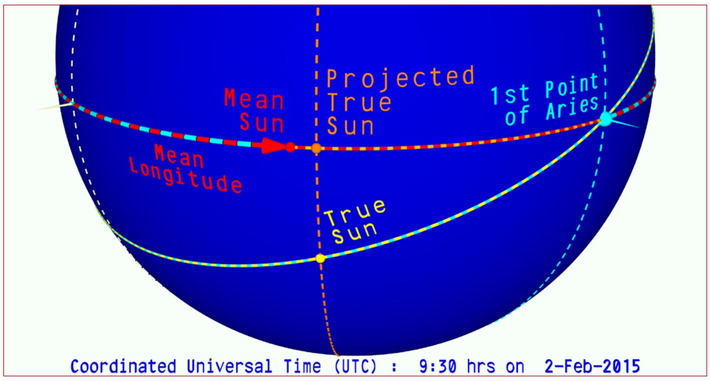

Fig. 2 Since out civil time-keeping system is tied to the diurnal rotation of the Earth, we have chosen the position of a 'ficticious' Mean Sun on the Celestial Equator as our primary civil time keeping system. We can calculate its position - the Mean Longitude, since it is connected to GMST (see Part 1). The Mean sun rotates around the Celestial Equator once per year.

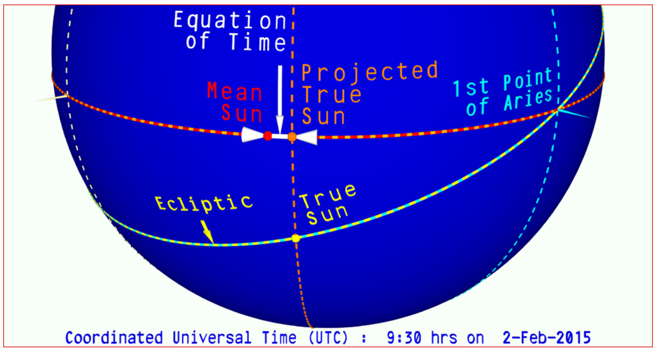

Fig. 3 The difference between Mean Longitude and Right Ascension is the Equation-of-Time.

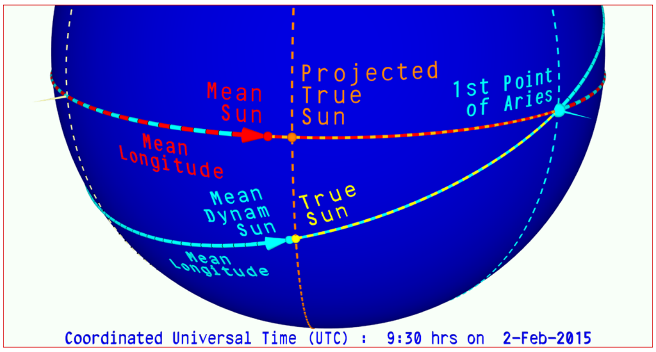

Fig. 4 It is necessary to invoke the Dynamical Mean Sun, another fictious Sun: this time on the Ecliptic. It is. It rotates uniformly around the ecliptic, once per year (as does the Mean Sun). Thus, its position is also defined by the Mean Longitude - but measured along the Ecliptic.

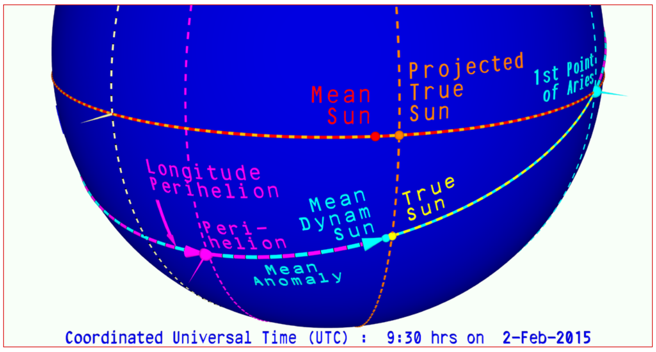

Fig. 5 The dynamics of the elliptical movement of the True Sun is tied to Perihelion - when the sun is closest to the Earth. The Longitude of Perihelion (origin 1st Point of Aries) is an astronomically known fact. The Mean Longitude is equal to Longitude of Perhelion + the Mean Anomaly.

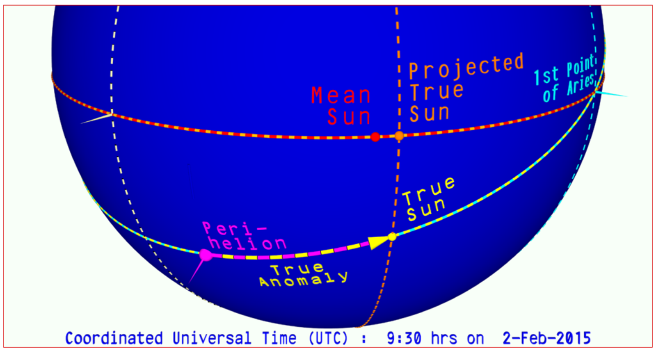

Fig. 6 Keplarian physics allows the True Anomaly, which is the position of the True Sun with respect to Perihelion, to be calculated in terms of the Mean Anomaly.

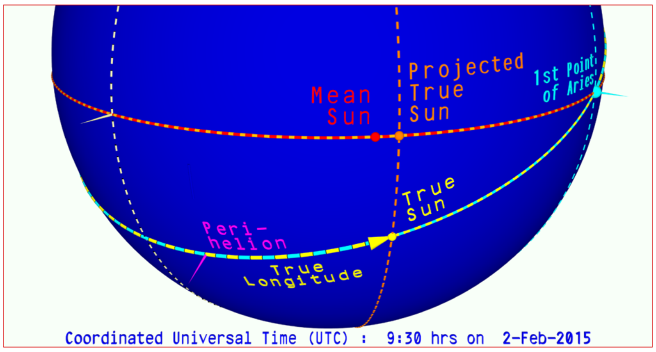

Fig. 7 Adding the True Anomaly to the Longitude of Perihelion yields the True Longitude of the Sun.

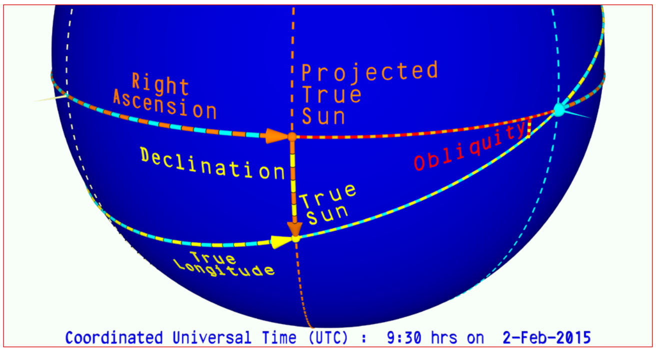

Fig. 8 Spherical Trigonometry, involving the True Longitude and the Obliquity, yields both Right Ascension & Declination

Calculating the True Sun's Longitude

This calculation for any given instant relies on three facts...

1 the Longitude of Mean Perihelion see Note 2 - ω - when the Earth is closest to the Sun, which corresponds to a date around 3rd January. This value is, once more, not exactly constant. Perihelion is moving towards the Vernal Equinox at the rate of 0.17° per century. For convenience, we will use... See Note 1

$$\omega^{deg} = 248.545\,360 + 0.017196 \times \text{YYYY} \text{ .... Eqn. 2.2}$$

where YYYY is the year

2 the Sun's apparent orbit is an ellipse - Kepler's First Law - with eccentricity - e - of 0.016 713. This value is not actually constant, but varying marginally... See Note 1

$$e = 0.017585 - 0.438 \times \left(\text{YYYY} / 1\,000\,000\right) \text{ .... Eqn. 2.3}$$

3 the apparent movement of the Sun obeys Kepler's Third Law - that a line joining the Earth to the Sun will sweep out equal areas in equal times.

This calculation requires the introduction of some new concepts and some very old mediaeval terms. Whereas we have used the 1st Point of Aries as our prime celestial origin, for elliptical orbits, we use instead the direction of Perihelion, when the Sun is closest to the Earth. Refer to Fig. 9, which is in the plane of the Ecliptic, unlike those illustrations in Part 1 of the series, which are in the plane of the celestial equator. For illustrative clarity, this shows an elliptical orbit of eccentricity of 0.4. The true value is a minute 0.0175, which if used in the diagram would make the elliptical path visually indistinguishable from a circle

Note the following...

- the *Earth*, at the centre of the illustration
- the *True Sun*, travelling on an ellipse, with the Earth at one of the ellipse's foci. Its position in relation to Perihelion - when the sun is closest to the earth - is called the *True Anomaly* - λ
- the imaginary *Mean Dynamical Sun* on the Celestial Ecliptic, a circle centred on the Earth. This body uniformly travels around the Ecliptic once in a tropical year. It is coincident with the Mean (equatorial) Sun at the 1st Point of Aries. It is thus the exact equivalent to the Mean Sun (on the Equator). Importantly, referenced to the 1st Point of Aries, at any moment in the year, its longitude on the Ecliptic is identical to the longitude of the Mean Sun on the Equator. Hence it can be calculated in terms of GMST. Its position in relation to perihelion is called the *Mean Anomaly* - M
- the imaginary *Eccentric Sun*, travelling on a circular path, whose centre is the centre of the ellipse, such

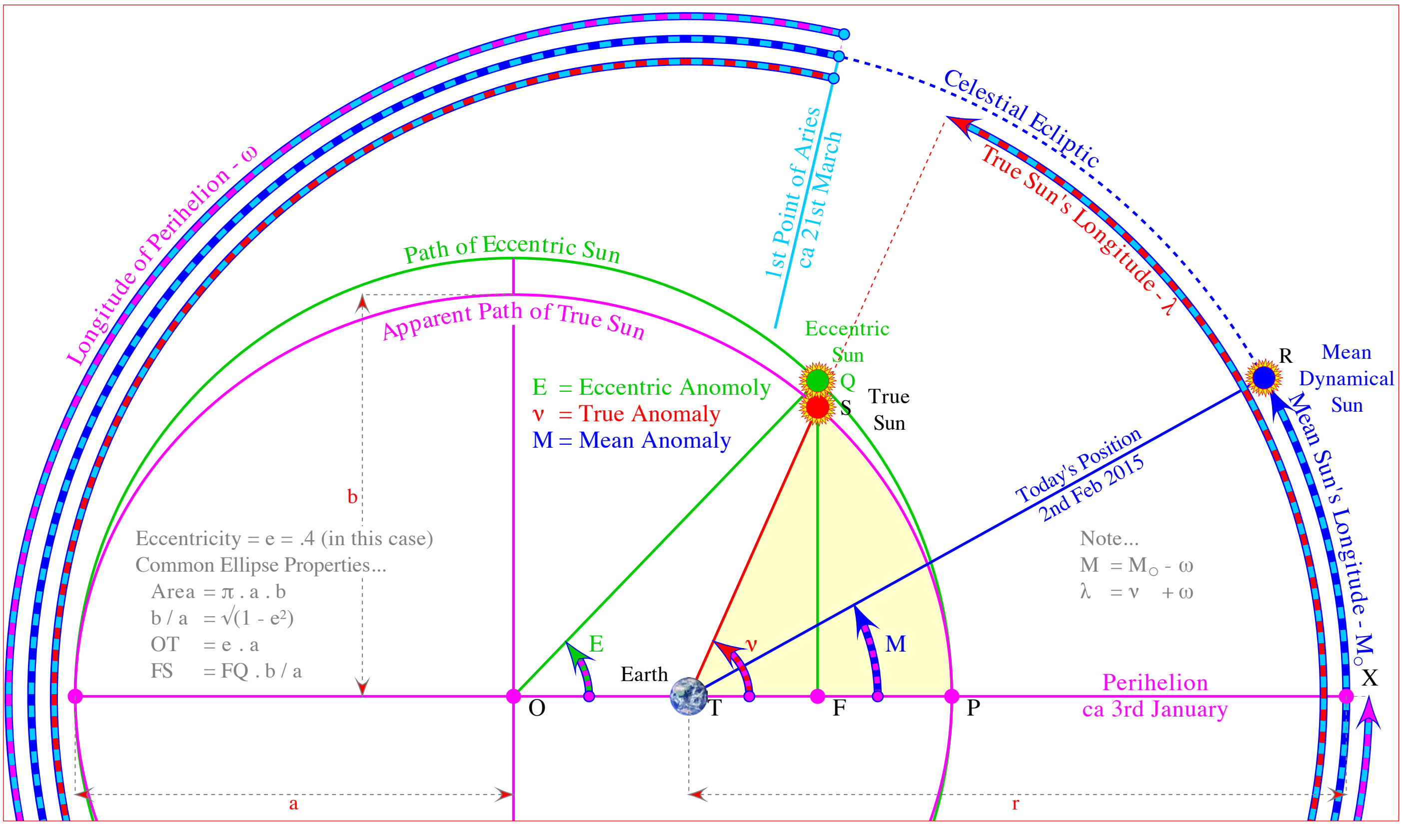

Fig. 9. True, Dynamical & Eccentric Suns, viewed in the ***Ecliptic*** *Plane*

that it is vertical (in the picture) above/below the True Sun. The Eccentric Sun is another imaginary body, which is only required as an intermediate to solve Kepler's Third Law. Its position in relation to perihelion is called the *Eccentric Anomaly* - E.

- the longitude of Perihelion - ω - provides the link between longitude and anomalies

$$M_0 \;=\; M + \omega \;\ldots\ldots\ldots\ldots\ldots\ldots\; \text{Eqn. 2.4}$$

$$\lambda \;=\; \nu + \omega \;\ldots\ldots\ldots\ldots\ldots\ldots\; \text{Eqn. 2.5}$$

Application of Kepler's third law reveals the connection between the Eccentric Anomaly, and the Mean Anomaly is...

$$M^{rad} \;=\; E^{rad} - e \times \sin\left(E^{rad}\right) \;\text{.. Kepler's Formula .. Eqn. 2.6}$$

Appendix 1 provides the derivation of this equation in the 17C method used before calculus was common. Unfortunately, Kepler's Formula - combining an angle E together with its trigonometrical sine - is not directly soluble. It requires an iterative solution. Application of a Newton Raphson approximation shows that - since the eccentricity of the ellipse is so near to zero - only one single iteration is required, to give the value of E

$$E^{rad} = M^{rad} - \left(e \times \sin\left(M^{rad}\right)\right) \Big/ \left(e \times \cos\left(M^{rad}\right) - 1\right) \ldots \text{ Eqn. 2.7}$$

Appendix 2, Figs 17 to 22 provides the derivation of this equation

The True Anomaly is connected to the Eccentric Anomaly by trigonometry...

$$\nu = \operatorname{atan2}\left(\sqrt{(1 - e^2)} \times \sin(E),\ \left(\cos(E) - e\right)\right) \ldots\ldots \text{ Eqn. 2.8}$$

Appendix 2, Fig. 23 provides the derivation of this equation. There is an alternate often quoted formula, see Note 3.

Calculating the Right Ascension & Declination

Knowing the Sun's Longitude and the Obliquity of the Ecliptic, it is simply a matter of solving a Spherical right angle triangle to find the Right Ascension & Declination.

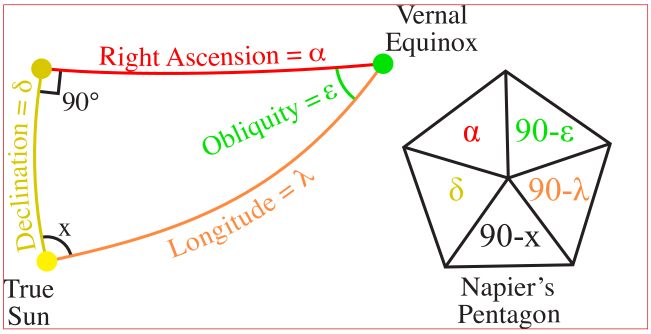

Fig. 10. Napier's Pentagon. The careful reader will note that this illustration does not conform to the others in this paper. As shown, δ would be calculted as a +ve number. Rest assured that the trigonometry works and using the conforming 360 - α & 360 - λ will provide a negative value of δ.

Obliquity in degrees is given by... See Note 1

$$\varepsilon^{deg} = 23.69930^{deg} - 0.00013 \times \text{YYYY} \ldots\ldots\ldots\ldots \text{Eqn. 2.9}$$

The right-angled triangle can be solved using Napier's pentagon which is a mnemonic aid that helps to find all relations between the angles in a right spherical triangle.

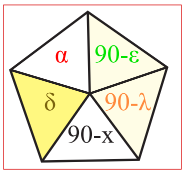

Fig. 11. Declination

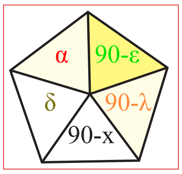

Fig. 12. Right Ascension

The mnemonic works thus... Write the six angles of the triangle (three vertex angles, three arc angles) in the form of a circle, sticking to the order as they appear in the triangle (i.e. start with a corner angle, write the arc angle of an attached side next to it, proceed with the next corner angle, etc. and close the circle). Then cross out the 90° corner angle and replace all angles non-adjacent to it by their complement to 90° (i.e. replace, say, λ by 90° − λ). The five numbers that you now have on your paper form Napier's Pentagon.

For any choice of three angles, one (the middle angle) will be either adjacent to or opposite the other two angles. Then Napier's Rules hold that the sine of the middle angle is equal to:

- the product of the cosines of the opposite angles, as in Fig. 11, thus...

  $$\sin(\delta) = \cos(90-\varepsilon)\times\cos(90-\lambda)$$

  $$\partial = \sin^{-1}\left(\sin(\varepsilon)\times\sin(\lambda)\right) \quad \text{................ Eqn. 2.10}$$

- the product of the tangents of the adjacent angles, as in Fig. 12, thus...

  $$\sin(90-\varepsilon) = \tan(\alpha)\times\tan(90-\lambda)$$

  $$\tan(\alpha) = \cos(\varepsilon)\times\tan(\lambda)$$

  $$\alpha = \operatorname{atan2}\left(\cos(\varepsilon)\times\sin(\lambda),\cos(\lambda)\right)$$

  .... Eqn. 2.11

Calculating the Local Hour Angle

All the astronomical calculations so far have related to Greenwich. In order to calculate the Sun's Altitude and Azimuth for an observer at a particular time of day and at a particular terrestrial location, we will require to find its Local Hour Angle...

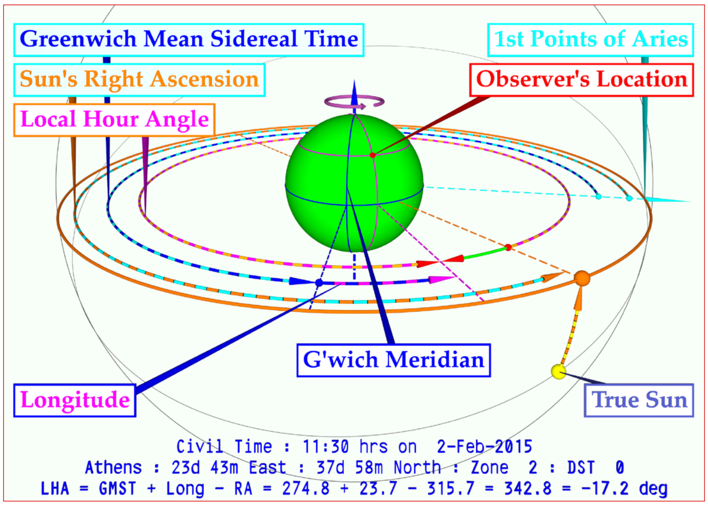

Fig. 13. Local Hour Angle

The Sun's Local Hour Angle is the angle between the Sun's meridian and the Observer's meridian.

At solar noon, the LHA is zero. Following normal practice, the LHA is negative before noon and positive after noon. In this document, however, it is counted positive from noon.

Looking at Fig. 13, we can deduce the connection between LHA - **h**^{0} -, Right Ascension - **α**^{0} - Greenwich Mean Sidereal Time - GMST - and the observer's longitude - **λ**$_t^0$. The LHA is the innermost dotted arc. The green arrow is 360^0 - LHA^0 (and is the 'normal' definition of LHA). Working from the outer arc, it is apparent that the Green arc =...

$$\alpha^{deg} - \lambda^{deg} - GMST^{deg} = 360^{deg} - h^{deg}$$

$$\therefore$$

$$h^{deg} = GMST^{deg} + \lambda^{deg} - \alpha^{deg} \quad \text{.................................Eqn. 2.12}$$

Calculating the Sun's Altitude and Azimuth

All the calculations so far in this paper have related to the Celestial Sphere. Now we must introduce the position of the observer at a given terrestrial Latitude & Longitude

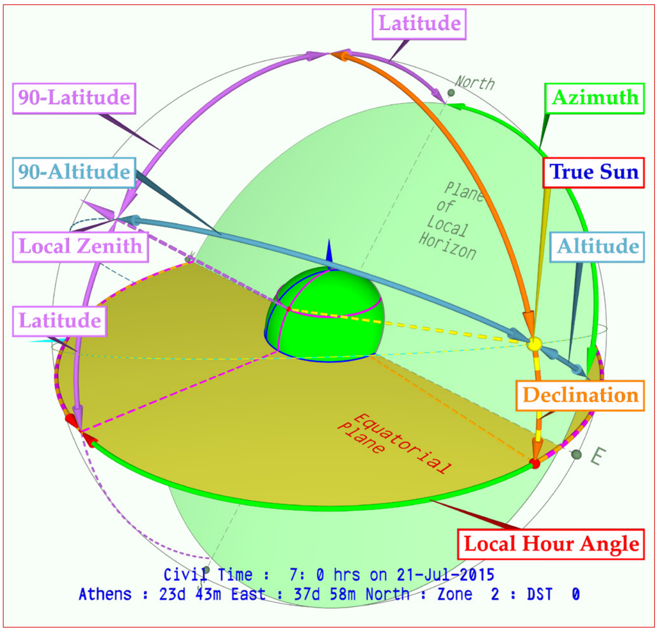

Fig. 14. The Equatorial Plane, the Horizonal Plane and the Observer

Fig. 14 shows the situation at a given time. Note the...

1) Equatorial Plane (olive coloured) from which are measured the...
   - Sun's declination (the orange arcs) - already calculated
   - Observer's Latitude (the purple arcs) - known
   - Observer's location with respect to the Sun: the Local Hour Angle (the red arc) - already calculated

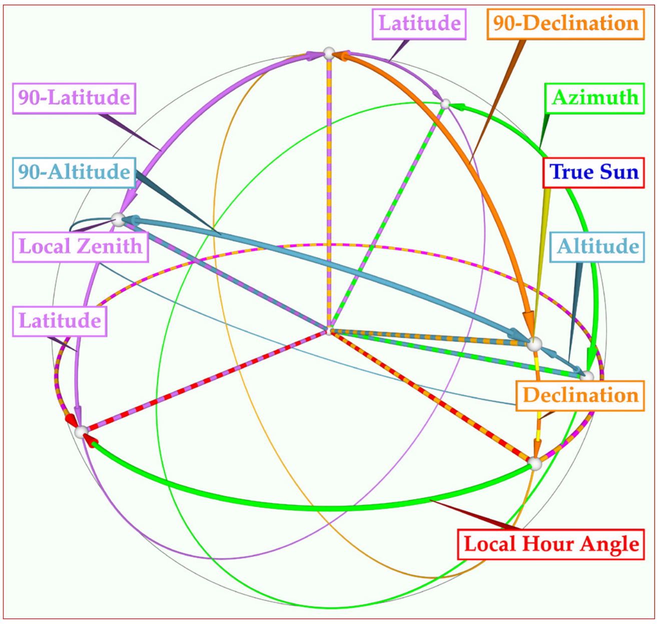

Fig. 15. As Fig. 13. but with extraneous information removed

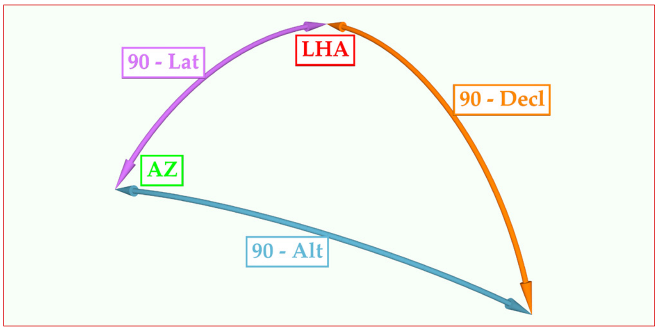

Fig. 16. The essential spherical triangle

2) Observer's horizontal plane (greenish coloured), from which is measured ...
   - Sun's Altitude (the blueish arcs) - to be found
   - Sun's Azimuth (the green arc) - to be found

Fig. 15 strips away extraneous detail to show the spherical triangles involved. While Fig. 16 shows the final spherical triangle to be solved.

In spherical trigonometry $^{\text{see Ref. 1}}$, the spherical laws of cosines and sines state that...

$$\cos(c) = \cos(a) \times \cos(b) + \sin(a) \times \sin(b) \times \cos(C)$$

$$\sin A / \sin(a) = \sin(B) / \sin(b) = \sin(C) / \sin(c)$$

.........Eqn. 2.13

where a, b & c are the angular arc lengths, while A is the angle between arcs b & c, etc. Applying the cosine law to Fig. 16, twice...

$$\cos(90 - Alt) = \cos(90 - Lat) \times \cos(90 - Decl) + \ldots$$
$$\sin(90 - Lat) \times \sin(90 - Decl) \times \cos(h)$$

.... Eqn. 2.14

and

$$\cos(90 - Decl) = \cos(90 - Lat) \times \cos(90 - Alt) + \ldots$$
$$\sin(90 - Lat) \times \sin(90 - Alt) \times \cos(Az)$$

.... Eqn. 2.15

Converting these to standard nomenclature gives the Sun's Altitude...

$$a^{rad} = \sin^{-1}\left(\sin(\phi) \times \sin(\delta) + \cos(\phi) \times \cos(\delta) \times \cos(h)\right)$$

.... Eqn. 2.16

$$\cos(A) = \left\{ \sin(\delta) \times \sin(\phi) \times \sin(a) \Big/ \cos(\phi) \times \cos(a) \right\}$$

.... Eqn. 2.17

Equation 26 provides some ambiguity to the azimuth value since (e.g.) the cosine of both 170° & 190° are the same. But if the sine law is applied..

$$\sin(A) = \cos(\delta) \times \sin(-h) / \cos(a)$$ .... Eqn. 2.18

then with no ambiguity, combining Eqn. 2.13 with Eqn. 2.14

$$A^{rad} = \operatorname{atan2}\left(\sin(A), \cos(A)\right)$$ .... Eqn. 2.19

Step 4 - Finding the Times of Sunrise and Sunset

Sunrise and Sunset are defined as the moment when the *apparent* centre of the Sun's disc is at zero altitude. In addition, the twilights are defined in terms of the *apparent* altitude of the centre of the Sun's disk

- civil twilight	altitude	0° to -6°
- nautical twilight:	altitude	-6° to -12°
- astronomical twilight:	altitude	-12° to -18°

Apparent altitude is the term used when the altitude is corrected for the effect of atmospheric refraction. The degree of refraction is dependent on the temperature and pressure of the atmosphere. There are empirical formulae allowing its estimation, which are presented - without comment - in steps 63 - 67 of Table 1, below. But See Chapter 16 of Ref. 2 for further elaboration. Refraction can be around ½° at altitudes close to zero in temperate climates. This is approximately equal to the angular size of the whole of the Sun's disc. One cannot find the moment of sunset without knowing atmospheric conditions and then iterating through the refraction calculations.

For most gnomonists, it is sufficient to estimate in the following fashion...

- forget about refraction
- calculate the declination δ at midday
- calculate the longitude corrected gnonomical Equation of Time, $\text{EoT}_{\text{Local}}$ at midday
- put altitude = 0 into Eqn. 25.

This yields the sunrise/set hour angle to be...

$$h^{deg}_{Sunrise/set} = \pm\cos^{-1}\left(-\tan(\varphi) \times \tan(\delta_{Noon})\right) \times 180^{deg} \big/ \pi$$

.... Eqn. 2.20

Then, converting to hours & including the EoT, yields the time and azimuth of Sunrise and Sunset...

$$T_{Sunrise}^{hrs} = 12^{hrs} - \left(h_{Sunrise/set}^{deg} \big/ 15 \right) - EoT_{Local}^{hrs} \text{ .... Eqn. 2.21}$$

$$T_{Sunset}^{hrs} = 12^{hrs} + \left(h_{Sunrise/set}^{deg} \big/ 15 \right) - EoT_{Local}^{hrs} \text{ .... Eqn. 2.22}$$

$$A_{Sunrise/set}^{deg} = \pm\cos^{-1}\left(-\sin\left(\delta_{Noon}\right) \big/ \cos\left(\phi\right) \right) \times 180^{deg} \big/ \pi$$

$$\text{...............Eqn. 2.23}$$

In passing, we may note that, adding together Eqns. 2.17 & 2.18, gives

$$EoT_{Local}^{mins} = 30 \times \left(T_{Sunset}^{hrs} + T_{Sunset}^{hrs} - 24 \right) \text{ ....Eqn. 2.24}$$

which means that, if you read the time of sunrise and sunset from your *local* newspaper, you can find the latitude corrected Equation of Time for your location. This was a trick used from Victorian times [See Ref. 3 & Note 4]. Since sunrise and sunset are usually only quoted to the nearest minute, it is somewhat surprising that this somewhat crude method gives the Equation of Time accurate to +/- 1 minute throughout the year in temperate latitudes.

Worked Example

Table 1, below, consolidates all the calculations in Parts 1 and 2. The functions that are used are given at then end of this section. Note carefully, that in some applications, these functions may not be present or called in a different manner.

In the Table above, the columns are…

i line number.
ii name of parameter
iii the parameters symbol, with a qualifier subscripted and its units superscripted, thus $EoT_{Gnomical}^{Min}$
iv the worked example resulting value
v the required formulae
vi the Equation number from the text - numbers thus 1.nn relate to Part 1 of this series, 2.nn to this part.

Where figures are given in red bracketed italics these are the results of working this example through a precision astronomical program, [See Ref. 4].

Input Observer's Location					
1	Longitude +ve East of Greenwich	λ_t °	23.71667	The Acropolis, Athens	
2	Latitude +ve North of Equator	φ °	37.96667		
3	Time Zone +ve East of Greenwich	TZ^{hrs}	2		
Input Observer's Date & Civil Time - (that is the Time that one reads on a clock or hears on the radio)					
4	Summer Time	DST^{hrs}	0		
5	Year	YYYY	2015	11:30 a.m 2nd February 2013	
6	Month	MM	2		
7	Day	DD	2		
8	Hour	HH	11		
9	Minute	MM	30		
Time related Parameters, Greenwich Mean Sidereal Time & the Sun's Mean Longitude					
10	UTC Uncorrected	UTC_{uncorr}^{hrs}	9.5	$ST^{hrs} - TZ^{hrs} - DST^{hrs}$	1.2
11	UTC Corrected	UTC^{hrs}	9.5	$\mathrm{mod}(UTC_{uncorr}^{hrs}, 24)$	
12		UTC °	142.5	$15 \times UTC^{hrs}$	
13	temporary value aaa is the correction to be made if the local date differs from the date at Greenwich	aaa	0	a = 0	1.3
14			0	if $(UTC_{uncorr}^{hrs} < 0)$ a = −1	
15			0	if $(UTC_{uncorr}^{hrs} > 24)$ a = +1	
16	temporary value	bbb	8973.5	367 × YYYY − 730 531.5	1.6
17	temporary value	ccc	3526	int({7. × int(YYYY + [MM + 9] / 12)} / 4)	
18	temporary value	ddd	63	int(275 × MM / 9) + DD	
19	Days since Midnight	D_{today}^{days}	0.39583	$UTC^{hrs} / 24$	
20	Days to 0:00 am since Epoch	$J2000^{days}$	5510.5	aaa + bbb − ccc + ddd	
21	Julian Centuries$_{2000}$	$T^{Jul\ Cent}$	0.15088	D_{2000}^{days} / 365 25	
22	Days to Now since Epoch	D_{2000}^{days}	5510.89583	$J2000^{days} + D_{Today}^{days}$	
23	Greenwich Mean Sideral Time	$GMST^{hrs}$	18.31737 *(18.31737)*	mod(6.697374558 + 0.065 709 824 419 08 × $J2000^{days}$ + 1.002 737 909 35 × UTC^{hrs} + 0.000 026 × $T^{Jul\ Cent\ 2}$, 24)	1.4
24		GMST °	274.76059	$GMST^{hrs}$ x 15	
25	Sun's Mean Longitude	M_O °	312.26059	mod{ (GMST ° − 180 ° − UTC °), 360 ° }	1.7
26		M_O^{rad}	5.44998	M_O ° × π / 180	

Table 1 Part 1

Astronomical Facts					
27	Perihelion Longitude	ω°	283.19530	$248.545\,36 + 0.017\,196 \times YYYY$	2.2
28		ω^{rad}	4.94269	$\omega^{\circ} \times \pi / 180^{\circ}$	
29	Eccentricity	e	0.01670	$0.017\,585 - 0.438 \times YYYY / 1{,}000{,}000$	2.3
30	Obliquity	ε°	23.43735 *(23.43758)*	$23.699\,3 - 0.000\,13 \times YYYY$	2.8
31		ε^{rad}	0.40906	$\varepsilon^{\circ} \times \pi / 180^{\circ}$	
Solving Kepler’s Theorem & Sun’s True Longitude					
32	Mean Anomaly	M^{rad}	0.50729	$M_O^{rad} - \omega^{rad}$	2.4
33	Eccentric Anomaly	E^{rad}	0.51552	$M_O^{rad} - \sin(M_O^{rad}) / \{\cos(M_O^{rad}) - 1 / e\}$	2.7
34	2nd iteration for example only >		0.51552	$E^{rad} - [M - E^{rad} + e \times \sin(E^{rad})] \div [e \times \cos(E^{rad}) - 1]$	-
35	True Anomaly	ν^{rad}	0.52381	$2 \times \mathrm{atan}\{\tan(E^{rad} / 2) \times \sqrt{[(1 + e) / (1 - e)]}\}$	2.8
36	Sun’s True Longitude	λ^{rad}	5.46650	$M_O^{rad} + \omega^{rad}$	2.5
37		λ°	313.20765 *(313.70149)*	$\lambda^{rad} \times 180^{\circ} / \pi$	
Sun’s Declination, Right Ascension & the Equation of Time					
38	Sun’s Declination	δ^{rad}	-0.29413	$\mathrm{asin}\{\sin(\varepsilon^{rad}) \times \sin(\lambda^{rad})\}$	2.10
39		δ°	-16.85245 *(-16.85158)*	$\delta^{rad} \times 180 / \pi$	
40	Sun’s Right Ascension	α^{rad}	-0.77365	$\mathrm{atan2}\{\cos(\varepsilon^{rad}) \times \sin(\lambda^{rad}), \cos(\lambda^{rad})\}$	2.11
41		α°	315.67321	$\mathrm{mod}(\alpha^{rad} \times 180 / \pi , 360)$	
42		α^{hrs}	21.04488 *(21.04468)*	$\alpha^{\circ} / 15$	
43	Equation of Time	EoT°	-3.41262	$GMST^{\circ} - \alpha^{\circ} - UTC^{\circ} + 180^{\circ}$	1.9
44		EoT_{Astro}°	-3.41262	$\mathrm{if}(EoT_X^{\circ} < -180^{\circ})\ EoT_{Astro}^{\circ} = EoT^{\circ} + 360^{\circ}$	
45			-3.41262	$\mathrm{if}(EoT_X^{\circ} > +180^{\circ})\ EoT_{Astro}^{\circ} = EoT^{\circ} - 360^{\circ}$	
46		$EoT_{Gnomical}^{\circ}$	3.41262	$- EoT_{Astro}^{\circ}$	1.10
47		$EoT_{Gnomical}^{min}$	13.65049 *(13.63333)*	$4 \times EoT_{Gnomical}^{\circ}$	
48	Longitude Correction	σ°	-6.28333	$LON^{\circ} - TZ^{hrs} \times 15$	1.11
49		σ^{min}	-38.84648	$\sigma^{\circ} \times 4$	
50	EoT Longitude Corrected	EoT_{Local}^{min}	-38.78381	$EoT_{Gnomical}^{min} + \sigma^{min}$	1.12
The Sun’s Altitude & Azimuth					
51	Observer’s True Hour Angle	h°	342.80405 *(342.80778)*	$\mathrm{mod}\{(GMST^{\circ} + \lambda_t^{\circ} - \alpha^{\circ}), 360\}$	2.12
52		h^{rad}	5.98306	$h^{\circ} \times \pi / 180^{\circ}$	
53	Observer’s Latitude	φ^{rad}	0.66264	$\varphi^{\circ} \times \pi / 180^{\circ}$	-
54	Sun’s Altitude	a^{rad}	0.57333	$\mathrm{asin}\{\sin(\varphi^{rad}) \times \sin(\delta^{rad})$ $+ \cos(\varphi^{rad}) \times \cos(\delta^{rad}) \times \cos(h^{rad})\}$	2.16
55		a°	32.84937	$a^{rad} \times 180^{\circ} / \pi$	
56	Sun’s Zenith Distance	z°	57.15063 *(57.01570)*	$90^{\circ} - a^{\circ}$	-
57	Sun’s Azimuth	sinA	0.33680	$\cos(\delta^{rad}) \times \sin(-h^{rad}) / \cos(a^{rad})$	2.18
58		cosA	-0.94158	$(\sin(\delta^{rad}) - \sin(a^{rad}) \times \sin(\varphi^{rad}))$ $/ (\cos(a^{rad}) \times \cos(\varphi^{rad}))$	2.17
59		A^{rad}	2.79808	$\mathrm{atan2}(\sin A, \cos A)$	2.18
60		A°	160.31807 *(160.32)*	$\mathrm{mod}(A^{rad} \times 180^{\circ} / \pi, 360^{\circ})$	

Table 1 Part 2

The Refraction Correction for the Sun’s Altitude - these are empirical formulae, see Ref. 2, they are not detailed in the text.					
61	Input Temperature	$T^{°C}$	20	Input	-
62	Input Atmospheric Pressure	$P^{millibars}$	1020		
63	Refraction Correction	$R°$	0.02389	if $(a° > 15°)$ $R° = 0.004\,52 \times \tan(z° \times \pi / 180) \times P^{millibars} / (273 + T^{°C})$	-
64			n.a.	if $(a° < 15°)$ $R° = P^{millibars} \times (0.1594 + 0.0196 \times a° + 0.00002 \times a°^{2}) / \{(273 + T^{°C}) \times (1 + 0.505 \times a° + 0.084\,5 \times a°^{2})\}$	
65	Sun’s Altitude Corrected	$a_{Corr}°$	32.82548	$a° - R°$	
Approximate Sunrise & Sunset					
66	Local Hr Ang, Sunrise/set	$h_{sr/ss}°$	76.32696	$\mathrm{acos}[-\tan(\varphi^{rad}) . \tan(\delta_{Noon}{}^{rad})] \times 180 / \pi$	2.20
67	Time of Sunrise	$h_{sr}{}^{hrs}$	7.55077 *(7.48333)*	$12 - (h_{/ss}° / 15) - EoT_{Local}$	2.21
68	Time of Sunset	$h_{ss}{}^{hrs}$	17.73486 *(17.81667)*	$12 + (h_{/ss}° / 15) - EoT_{Local}$	2.22
69	Sun’s Azimuth at Sunrise	$A_{sr}°$	111.57578 *(111)*	$\mathrm{acos}[\sin(\delta_{Noon}{}^{rad})/\cos(\varphi^{rad})] \times 180 / \pi$	2.23
70	Sun’s Azimuth at Sunset	$A_{ss}°$	248.42422 *(249)*	$360° - A_{sr}°$	

Table 1 Part 3

Functions that are used in the Table are...

- **degrees** & **radians** function - may be replaced by × 180 / π or by × π / 180
- trigonometric functions, **sin**, **cos** & **tan**. In most implementations, these require input in radians: while the inverse functions asin, acos, atan output in radians. If this is not the case, many of the degree/radian conversions below can be ignored - but not in Steps 34-37, where radians must be used. Note that in traditional trigonometry **asin** was written as **sin^{-1}**.
- **atan2** function - this now exists in most programming languages and returns the inverse tangent function in the correct quadrant, but requires both an x and y input parameter. Irritatingly, while most scientific languages implement this as the more trigonometrically correct atan2(y,x), Microsoft Excel uses atan2(x,y).
- **int** function - this simply strips the fractional part of a number away. Note, once more, that most scientific languages implement this strictly for positive & negative number. Thus int(1.6) = 1 and int(-1.6) = -1, but once more Microsoft Excel differs: int(1.6) = 1 but int(-1.6) = -2. This difference in not of interest below, since the int function operates only on positive numbers
- **mod** function. Particularly in angular calculations, this reduces a number to lie in a particular range (e.g. from 0° to 360°). Thus mod(370°,360°) = 10° = mod(-350°, 360°). Some languages make this function into an arithmetic operator: thus, in Python, 370 % 360 = 10.

Accuracies

In the calculations above, the only non-derived astronomical parameters used are the...

- length of the tropical year,
- eccentricity of the Earth’s orbit,
- obliquity of the Ecliptic,
- longitude of perihelion,
- a single factor covering precession.

With this small coterie of values, it is perhaps remarkable that a relatively simple (if long) approach can yield the accuracies stated over a period of 50 years.

GMST	+/- 0.00 ^{secs}
Right Ascension	+/- 3 $^{secs\ of\ time}$
Declination	+/- 18 $^{secs\ of\ arc}$
Equation of Time	+/- 2.2 $^{secs\ of\ time}$
Altitude	+/- 0.7 $^{minutes\ of\ arc}$
Azimuth	+/- 1.3 $^{minutes\ of\ arc}$

The stated accuracies have been derived with reference to 75,000 calculations using the 2012 edition of the US Naval Observatory’s MICA program $^{see\ Ref.\ 4}$.

The above calculations are more than sufficient for most gnomonists. However, if one wishes to pursue the calculations to a greater degree of accuracy. There are a number of factors that have to be considered

- The slowing of the year’s rotation, as seen in the introduction of leap seconds in the calendar.
- The fact that solar dynamics use difference time and position reference frameworks.
- The ‘correct’ dynamical approach calculates the Earth’s longitude for a particular instant of time. Sunlight reaches the Earth some 8 minutes later. During this time the Earth has moved somewhat. This effect is called Aberration.
- We have calculated the Sun’s longitude about the Ecliptic and assumed that its latitude is zero. This is not quite true.

- We have ignored the “rattling and banging” of Nutation, which varies right ascension by up to 20 secs of arc and obliquity by up to 10 secs of arc. Nutation is caused by the gravitational pull of the Moon (& especially Jupiter) on the equatorial bulge of the Earth’s shape.
- Our calculations relate to the centre of the Earth. Our position on the surface of the Earth varies the values of both Right Ascension & Declination.

If the reader wishes to delve deeper, Ref. 4 provides a useful outline and Ref. 2 provides the greatest depth achievable without access to serious professional astronomical computing routines. The latter are available see Ref. 6, through the International Astronomical Union. However , their use by amateurs requires knowledge of Fortran or the “C” programming language.

Appendix 1 - Derivation of Kepler’s Law

Kepler’s Equation...

$$M^{rad} = E^{rad} - e\times\sin\left(E^{rad}\right) \text{ .... Eqn. 2.25}$$

is the result of his 1st and 2nd Laws of Planetary Motion

i The orbit of every planet is an ellipse with the Sun at one of the two foci.

ii A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

The means of developing this formula therefore demands that we can calculate the area swept out in any given time, e.g. from Perihelion. This is the yellow shaded area in Fig. 9. Finding this area can be done by simple means using an old technique,see Ref. 7. The steps required are shown in Figs 16 to 22 below.

The next step shown in Fig. 23, not quite so easy to grasp, relates to the “equal areas during equal intervals of time”. This indicates that the area just calculated is proportional to the area swept out by the Mean Dynamical Sun in the same period.

Finally, Fig. 24 shows how the true anomaly - ν - is related to the Eccentric Anomaly - E

Appendix 2 - Derivation of Newton Raphson approximation for Kepler’s Formula

Kepler’s Formula, Equn, 2.21, cannot be solved directly.

So an iterative solution must be sought. The Newton-Raphson method See Ref. 8 is an efficient method, provided that one can differentiate the function concerned. The method states that, if an estimation E_n is obtained, a better estimation E_{n+1} may be obtained, thus...

$$E_{n=1} = E_n - \frac{fn\left(E_n\right)}{fn^{/}\left(E_n\right)} \text{ .... Eqn. 2.26}$$

but , rewriting Eqn. 2.21 and differentiating...

$$fn(E_n) = M - E_n + e\times\sin\left(E_n\right) \text{ ....Eqn. 2.27}$$

$$fn^{/}(E_n) = e\times\cos\left(E_n\right) - 1 \text{ .... Eqn. 2.28}$$

To apply the Newton Raphson formula, we make a guess to start the process and then repeatedly put Eqns. 40 & 41 into Eqn. 40...

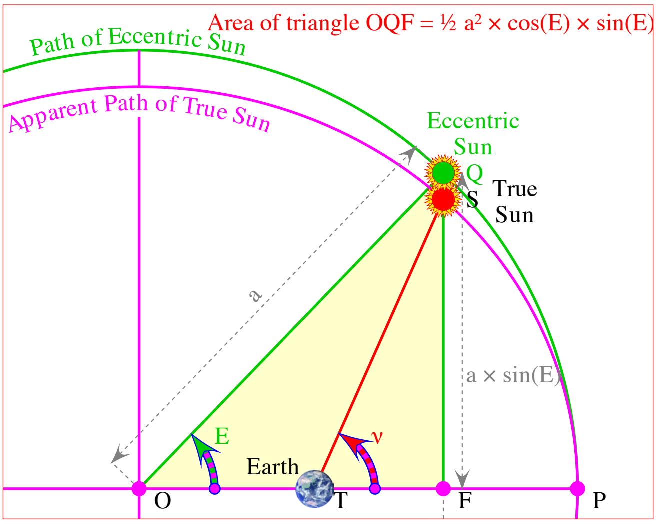

Fig. 17. Solving Kepler’s Formula - Step 1

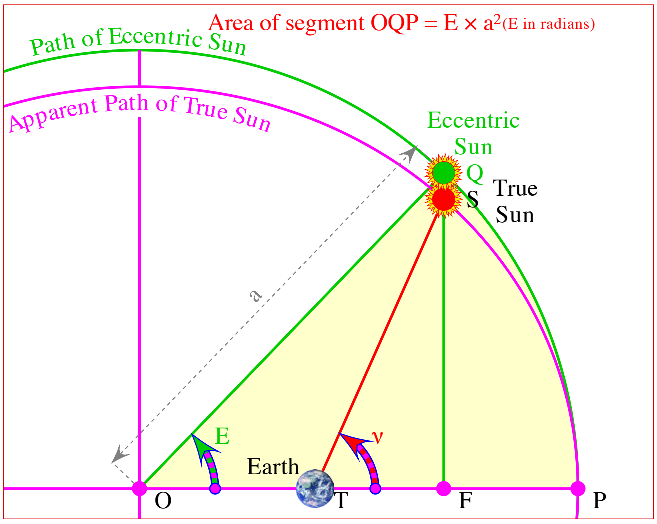

Fig. 18. Solving Kepler’s Formula - Step 2

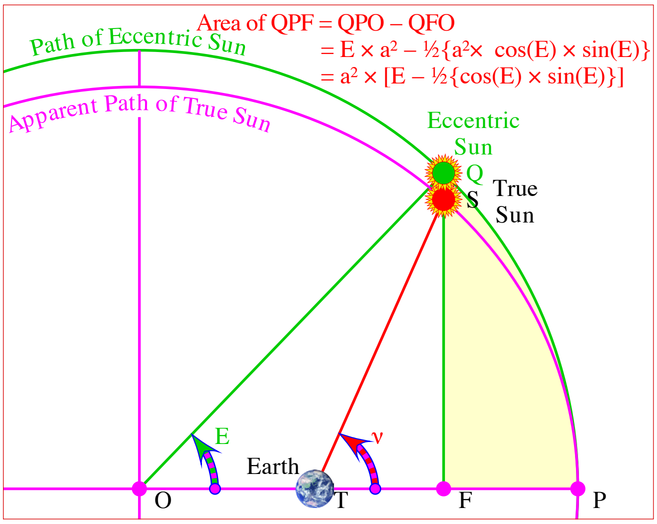

Fig. 19. Solving Kepler’s Formula - Step 3

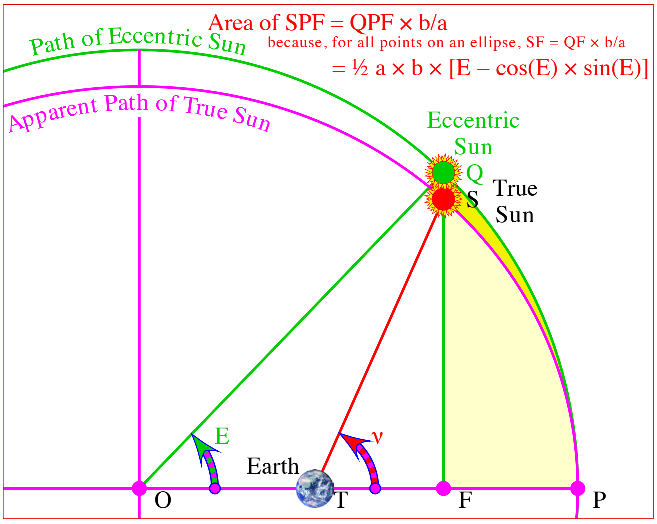

Fig. 20. Solving Kepler's Formula - Step 4

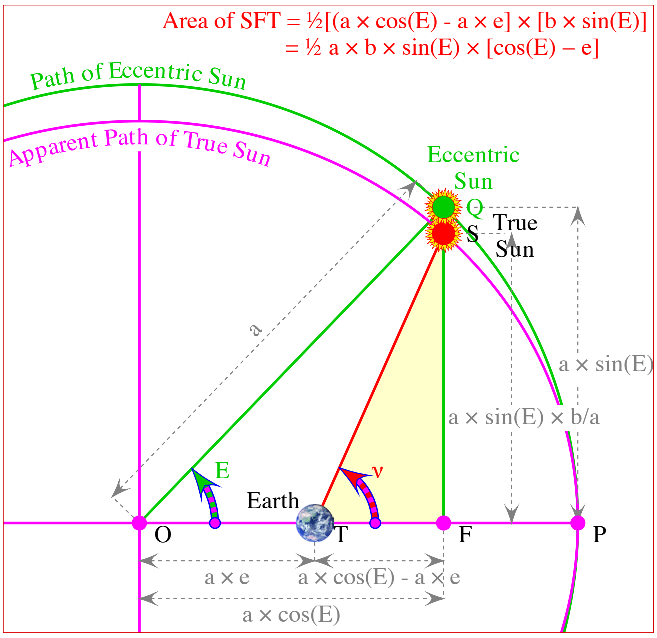

Fig. 21. Solving Kepler's Formula - Step 5

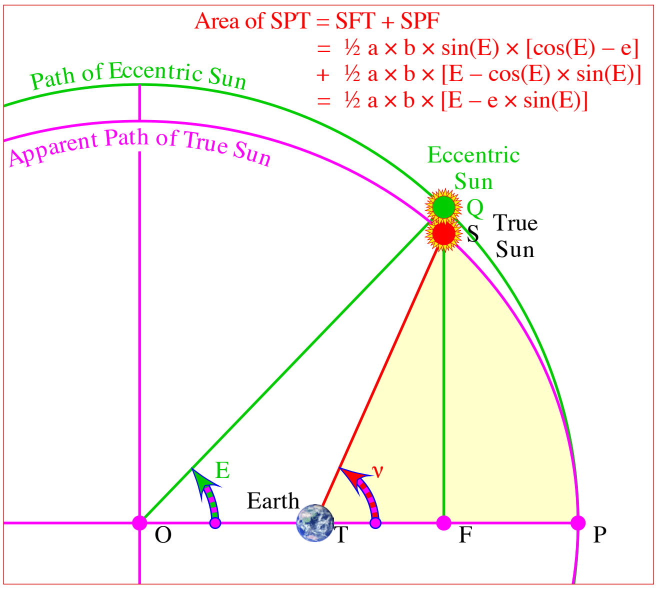

Fig. 22. Solving Kepler's Formula - Step 6.
The yellow segment is the area swept out by the True Sun

Fig. 23. Step 7 - see overleaf

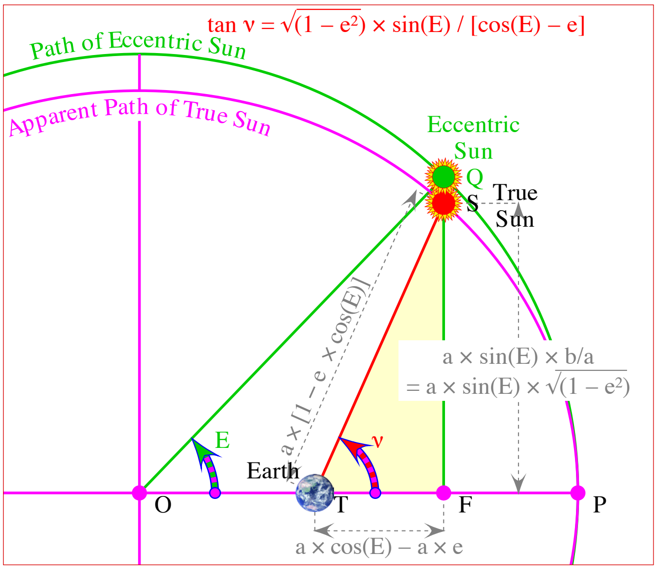

Fig. 24. Solving Kepler's Formula - Step 8

$$\begin{aligned} E_1 &= \text{a guess} \\ E_2 &= E_1 - \left(M - E_1 + e \times \sin(E_1)\right) / \left(e \times \cos(E_1) - 1\right) \\ E_3 &= E_2 - \left(M - E_2 + e \times \sin(E_2)\right) / \left(e \times \cos(E_2) - 1\right) \\ E_4 &= E_3 - \left(M - E_3 + e \times \sin(E_3)\right) / \left(e \times \cos(E_3) - 1\right) \end{aligned} \quad \text{........... Eqn. 2.29}$$

and we repeat the process until there is negligible difference between E_n and E_{n+1}

We make our first guess. as $E_1 = M$, then...

$$\begin{aligned} E_2 &= M - \left(M - M + e \times \sin(M)\right) / \left(e \times \cos(M) - 1\right) \\ &= M - \left(e \times \sin(M)\right) / \left(e \times \cos(M) - 1\right) \text{............. Eqn. 2.30} \end{aligned}$$

Since the eccentricity is so small, it transpires that this is the only iteration needed ! It is left to the reader to show that, for any value of M^{rad} between 0 and 2π, the difference between E_2 and E_3 is less than +/- .5 $^{\text{seconds of arc}}$, which is sufficiently precise for that which is required by the dialist. The difference between E_3 and E_4 is effectively zero.

Notes

1. Equations for Eccentricity, Obliquity & Longitude of Perihelion were adapted from the formulae quoted in the Astronomical Almanac [Ref. 10].
2. If one consults the Astronomical Almanacs over the years, the reader will note that the moment of Perihelion varies back & forth in an apparently random fashion between Jan 2 and Jan 5th as shown below...

2013	Jan 2, 06:38	2017	Jan 4, 16:18
2014	Jan 4, 13:59	2018	Jan 3, 07:35
2015	Jan 4, 08:36	2019	Jan 3, 07:20
2016	Jan 3, 00:49	2020	Jan 5, 09:48

The table given the moment the centre of the Earth is closest to the Sun. The mean value of Perihelion - ω, as given in the equations presented in this paper, is the moment when the centre of gravity of the Earth/Moon combination is closest to the Sun. This combined mass has its centre of gravity some 1700 kms below the Earth's surface - about ¼ of the way towards the Earth's centre. As far as Keplarian physics is concerned, the

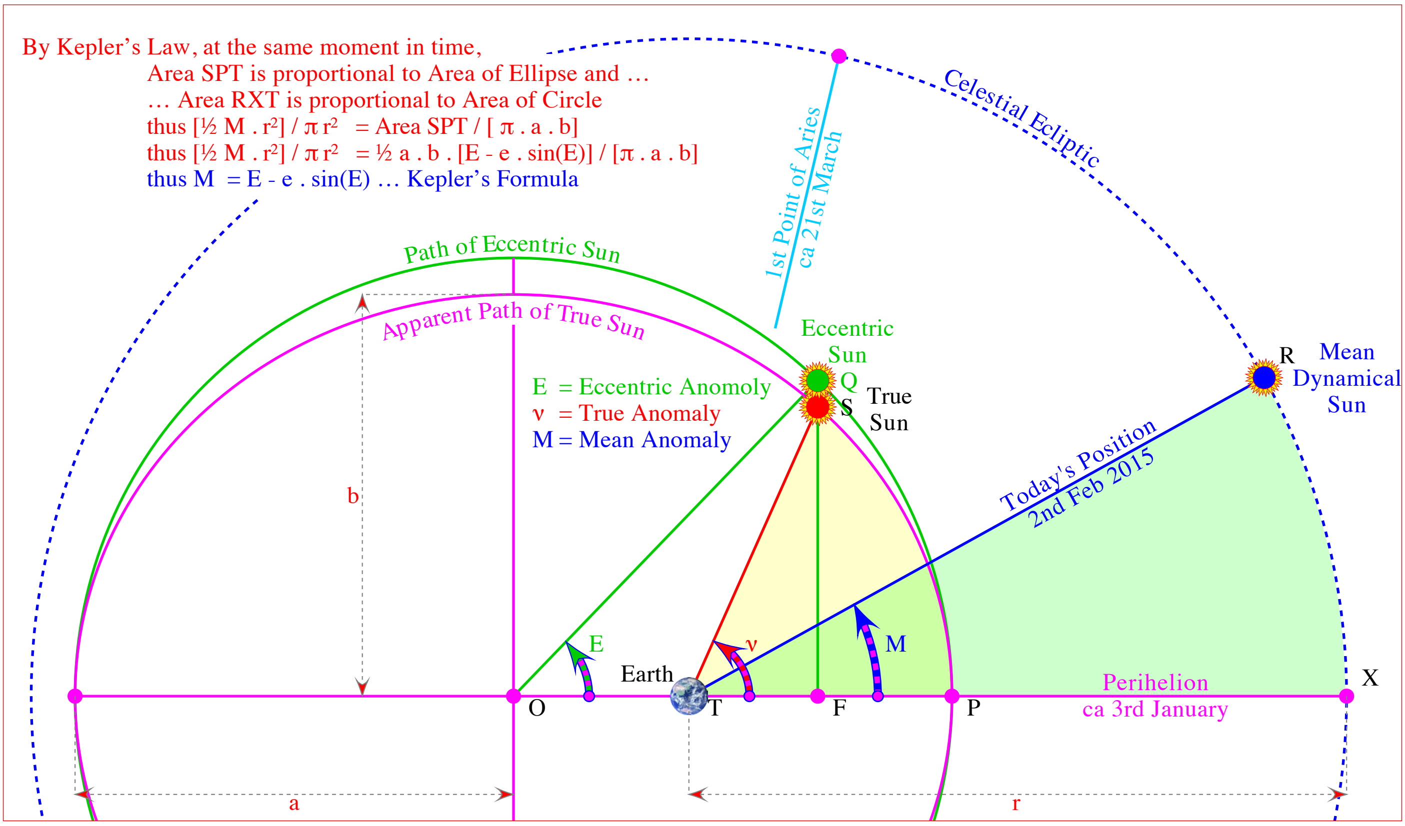

Fig. 23. Solving Kepler's Formula - Step 7.
The green segment is the area swept out by the Mean Sun. By Kepler's Law, this must equal the yellow segment.

calculations above relate to the unequal dumb-bell that is the Earth/Moon combination.

3. Eqn. 20 requires the atan2 function to provide an answer in the correct quadrant, (ν must be in the same quadrant as E). An alternate formulae is often published, which avoids the use of atan2, through the use of the trigonometric half-angle formulae...

$$\tan\left(\frac{\nu}{2}\right)=\tan\left(\frac{E}{2}\right)\times\sqrt{\left(\frac{(1+e)}{(1-e)}\right)} \text{ .... Eqn. 2.31}$$

The two are functionally identical. This formula can, with some cumbersome trigonometry, be derived from Eqn. 20.

4. Ref. 1 - below - gives the formula as...

$$2\times EOT = \text{Length of afternoon} - \text{Length of morning}$$

.... Eqn. 2.32

5.

Spike Milligan...
What's the Time, Eccles?
Wait, I've got it written down on a piece of paper...
... Eight o'clock.
Where did you get that?
I asked a man what the time was and he wrote it down for me. It's very nice because when people ask me the time, I can tell 'em because I've got it written down on a piece of paper.
What do you do when it's not eight o'clock?
I don't look.
So how do you know when it is eight o'clock?
I've got it written down on a piece of paper......

References

The reader is referred to the general References Part 1 of this series.

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Basic Solar Positional Astronomy
Part 3: Fourier Derived Formulae
KEVIN KARNEY

Preamble

If the above routines are too involved for easy use, one may always use Fourier deduced trigonometric series. This study was triggered by the author's interest in derivation and quality of the Equation of Time formula given in the BSS Glossary [Ref 1]

$$E_a^{\text{mins}} = \begin{pmatrix} -0.000\,00.75\ldots \\ -0.001\,868\cos(\omega)+0.032\,077\sin(\omega)\ldots \\ +0.014\,165\cos(2\omega)+0.040\,849\sin(2\omega) \end{pmatrix} \times {}^{720}\!/_{\pi}$$

$$\omega \quad = {}^{2\pi n_d}\!/_{365}$$

$$n_d \quad = 1 \text{ at noon on 1 Jan, 32 on 1 Feb, etc.} \qquad \ldots\ldots\ldots\ldots\text{Eqn 3.1}$$

If during leap years, 366 replaces 365 in the second line, this formula yields an accuracy of +48 & -36$^{\text{secs of time}}$ over the first 50 years of this century. The method described hereafter is thorough and produces results with far greater precision - for EoT, Declination and Right Ascension - than is generally required in dialing. For many, more simple formula will suffice: these are also deduced - providing some improvement over those provide in the Glossary.

The Fourier Approach

Any 'signal' that repeats with time (for example the Equation of Time, or Declination) can be approximated by the sum of a number of pure sine (or cosine) curves.

The theory states that an approximation of a function can be made...

$$f(x) \quad \approx \quad Av + \sum_{n=1}^{N} A_n \times \sin\left(n \times \left(\theta^{rad} + \varphi_n^{rad}\right)\right) \quad \ldots\ldots\ldots\ldots\text{Equ 3.2}$$

- Av = the average of the signal over an integer number of its periods
- n = harmonic number,
- A = amplitude of a particular harmonic
- θ = phase of that particular harmonic,
  (e.g. on 20th day of the year,
  θ = 2 x π x 20$^{\text{day}}$/365$^{\text{day}}$),
- φ = offset of the harmonic's zero point from start of computations,
  (e.g. offset of vernal equinox on 21 Mar from Jan 1
  φ ≈ 90$^{\text{day}}$/365$^{\text{day}}$ x 2 x π) ,

See Note [1] for alternative versions of Equ 3.2

The accuracy obtained by this approximation method depends on the number of harmonics that are chosen. Bretagnon and Simon [Ref 2] used 1080 terms to model the Sun's Longitude. Even, with just 6 terms, a saw tooth signal can be quite well modelled. See Fig 1. The method is widely used in many fields of industry - electronics, radio & seismic processing, to name but a few.

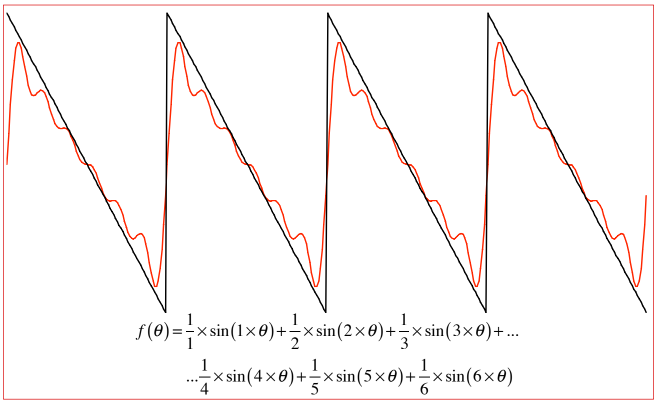


Fig. 1. This shows how even a linear periodic shape can be simulated by the sum of trigonometric components. More components, better fit.

A simple means of extracting the harmonic amplitudes and offsets from a 'signal' is illustrated in the Appendix. This method works very well if the duration of one repeating cycle is known - a 365 calendar year does not do well for most solar parameters, whereas a 365.25 cycle does better since it is closer to the length of a tropical year

However clever the Fourier approach may be in analysis, it does not cater well for the slow secular changes that are common in astronomy. As far as we are concerned, these relate to precession, the value of eccentricity, obliquity and perihelion longitude. These may by cyclical over the very long-term - but over our life time, their changes are effectively linear but small.

To overcome this problem, the following steps were followed...

1 input (EoT, Decl & RA) was calculated from MICA [Ref 3] every 6 hours over a period of some 50 years from noon on 1st Jan 2000 to midnight on 1st Jan 2051 - which is exactly 50 x 365.25 days

2. for each of the three input types, the values where divided in 50 x 365.25-day cycles, each containing 1461 values.

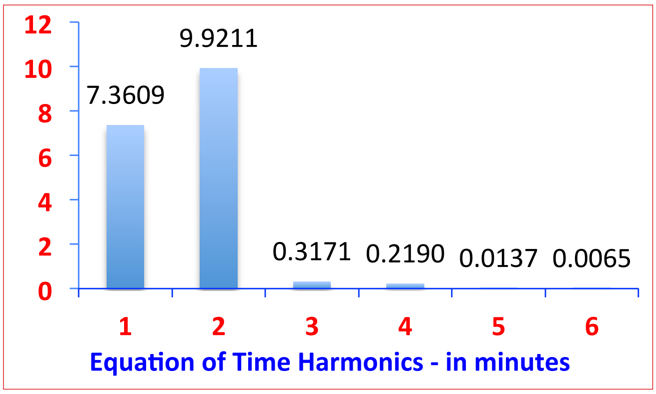


Fig. 2. The first 6 EoT harmonic amplitudes

3 for each of the 50 cycle, the first 6 harmonics were calculated using exactly the method described in Appendix 1

4 a Fourier approximation of the input was back-calculated using Equ 3.1 and compared with the input. In all cases, the last two harmonics provided little more than noise, so were discarded . See Fig 2

5 for the larger amplitude harmonics, the change in amplitude and offset was analysed over the 50 cycles and the value A_n and φ_n in Eqn 3.1 were replaced by the equation of their trend lines . For example see Fig 3. This shows how the second EoT harmonic amplitude varies over the 50 cycles and its linear trendline.

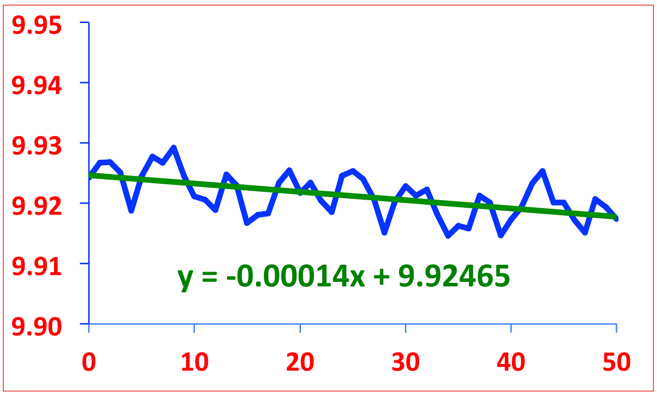


Fig. 3. The trend in the EoT 2nd harmonic amplitude over 50 years

The Phase Angle & Cycle

All the formula presented need to have the date/time converted into a phase angle - θ. Each uses the Days since the 2000 Epoch (D_{2000}^{days}) as input. These can be easily found using the routines down to lines 10-20 of Table 1. Thereafter the cyclical angle θ^{rad} is calculated thus...

$$
\begin{aligned}
bbb &= 367 \times YYYY - 730531.5 \\
ccc &= -int\left(\left(7 \times int\left(YYYY + (MM+9)/12\right)\right)/\ 4\right) \\
ddd &= int(275 \times MM/9) + DD \\
D_{today} &= (HH + MM/60)\ /\ 24 \\
D_{2000}^{days} &= bbb + cccc + ddd + D_{today} \\
Cycle &= int\left(D_{2000}^{days}\ /\ 365.25\right)
\end{aligned}
$$

Cycle & θ^{rad} is applied in each of the routines below.

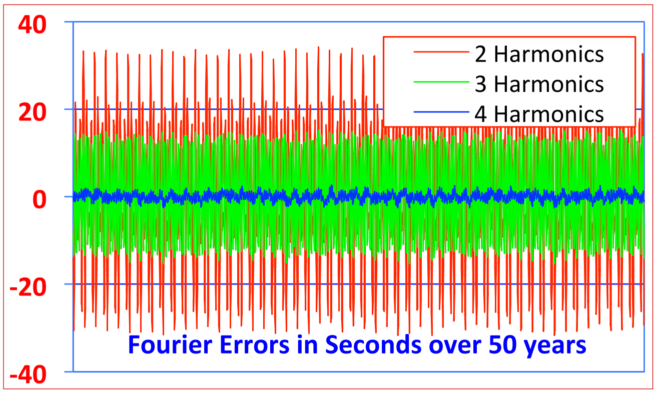


Fig. 4. 50 year trend in the EoT 2nd harmonic amplitude.

Equation of Time

The Equation of Time may be estimated thus:

$$
\begin{aligned}
Amp_1^{mins} &= 7.36303 - Cycle \times 0.00009 \\
Amp_2^{mins} &= 9.92465 - Cycle \times 0.00014 \\
\varphi_1^{rad} &= 3.07892 - Cycle \times 0.00019 \\
\varphi_2^{rad} &= -1.38995 + Cycle \times 0.00013 \\
EoT_1^{mins} &= Amp_1 \times \sin\left(1 \times \left(\theta + \varphi_1\right)\right) \\
EoT_2^{mins} &= Amp_2 \times \sin\left(2 \times \left(\theta + \varphi_2\right)\right) \\
EoT_3^{mins} &= 0.31730 \times \sin\left(3 \times \left(\theta - 0.94686\right)\right) \\
EoT_4^{mins} &= 0.21922 \times \sin\left(4 \times \left(\theta - 0.60716\right)\right) \\
EoT^{mins} &= 0.00526 + EoT_1 + EoT_2 + EoT_3 + EoT_4
\end{aligned}
$$

...........Eqn 3.4

This yields the Equation of Time to +/- 3 $^{\text{seconds of time}}$ from 2000 to 2050. Dropping the fixed and the fourth term (EoT_4) reduces the accuracy to +/- 16 $^{\text{seconds of time}}$.

Additional simplification of the above routine yields..

$$
\begin{aligned}
EoT^{mins} = {} & 7.36 \times \sin(\theta + 3.08) + \ldots \\
& 9.92 \times \sin(2 \times \theta - 2.78) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots Eqn.\ 3.5
\end{aligned}
$$

This has errors of +/- 34 $^{\text{seconds of time}}$, which makes it adequate for most gnomonical purposes.

1st harmonic overtone. The amplitude factor of 7.3630^{min} in the term EoT_1 primarily represents the eccentricity effect, which cycles once per year, with perihelion as origin. The offset angle of ...

$3.07892^{rad} = 176° = 176 \times 365.25/360^{days} = 179^{days}$ which is the time of mean aphelion after 1st Jan.

2nd harmonic overtone. The amplitude factor of 9.92465^{min} in the term EoT_2 represents the major component of the obliquity effect, which cycles twice per year, with the equinox as origin. The offset angle of ...

$1.38995^{rad} = 80° = 80 \times 365.25/360^{days} = 81^{days}$ which is the time of mean vernal equinox after 1st Jan.

3rd and 4th harmonic overtones. These are mostly due to the fact that the obliquity effect is essentially tangential rather than sinusoidal (see Equ 28)

The error bands for the 4, 3 & 2 harmonic estimations are given in Fig 4.

Declination

The analysis for Declination was more complex,. Fig 5 shows the plot of 1st harmonic amplitude against Cycle.

This shows a linear downward trend, together with a sinusoidal shape. To investigate this harmonic, first, the linear trend was extracted, leaving a normal sine curve. Second, this sine curve which was subject to another Fourier analysis, which showed an interesting 18-year recurrence, which means that it might be related to the Saros eclipse cycle. In turn, this suggest that it represents one of the lunar nutational effects, varying the

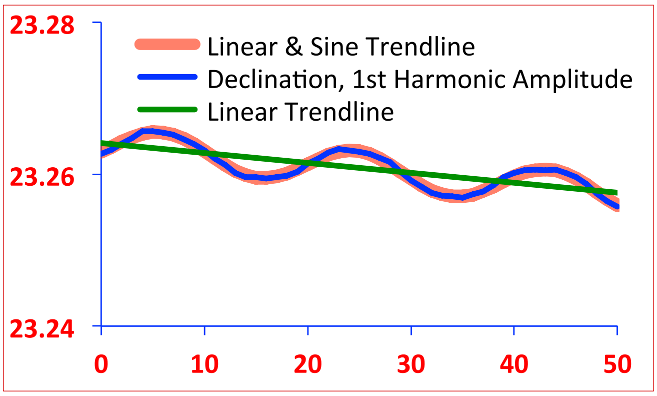


Fig. 5. The trend in the Declination 1st harmonic amplitude over 50 years

Earth's obliquity - and hence the declination. This is to be expected.

The Declination of the Sun may be estimated thus:

$$\begin{aligned}
Amp_1^{\deg} &= 23.2639 - Cycle \times 0.000131\ldots \\
&\quad \ldots+ 0.0024 \times \sin(Cycle \times 0.335 - 0.4) \\
\varphi_1^{rad} &= -1.38819 + Cycle \times 0.000135 \\
\delta_1^{\deg} &= Amp_1 \times \sin\left(1 \times \left(\theta^{rad} + \varphi_1^{rad}\right)\right) \\
\delta_2^{\deg} &= 0.380897 \times \sin\left(2 \times \left(\theta^{rad} - 0.720483\right)\right) \\
\delta_3^{\deg} &= 0.171178 \times \sin\left(3 \times \left(\theta^{rad} - 0.347175\right)\right) \\
\delta_4^{\deg} &= 0.008067 \times \sin\left(4 \times \left(\theta^{rad} - 0.272216\right)\right) \\
\delta^{\deg} &= 0.37657 + \delta_1 + \delta_2 + \delta_3 + \delta_4 \ldots\ldots\ldots\ldots\ldots \text{ Equ 3.6}
\end{aligned}$$

In the formula above, the first line of the equation shows the linear trend of Amp_1: the second line, the sinusoidal trend.

This yields Declination to +/- 30 $^{\text{seconds of arc}}$ from 2000 to 2050. Dropping the fourth term (δ4) reduces the accuracy to +/- 52 $^{\text{seconds of arc}}$. Dropping the third term (δ3) reduces the accuracy to +/- 11 $^{\textbf{minutes}\text{ of arc}}$.

Additional simplification of the above routine yields..

$$\begin{aligned}
\delta^{deg} &= 0.377 + 23.264 \times \sin(\theta - 1.388) \ldots \\
&\quad \ldots + 0.381 \times \sin(2 \times \theta - 1.44) \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{Eqn 3.7}
\end{aligned}$$

This has errors of +/- 21 $^{\text{minutes of arc}}$, which makes it adequate for most gnomonical purposes.

Right Ascension

Eqn 1.9, from Part 1 of this series, can be rearranged as follows...

$$\alpha^{hrs} = GMST^{\,hrs} - UTC^{hrs} + 12^{hrs} + EoT^{hrs}_{gnomonical} \ldots\ldots\ldots\text{Eqn. 3.8}$$

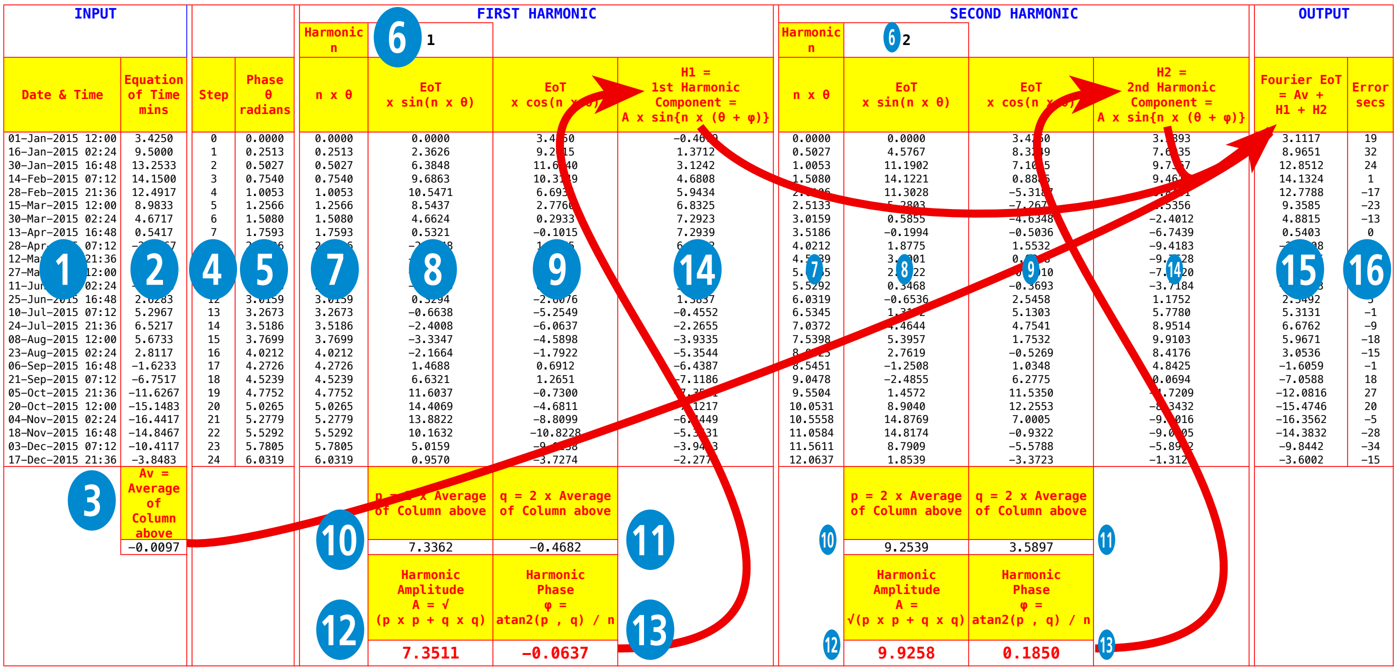
GMST, to the level of the accuracy of this study, is linear. Thus GMST - UTC +12 is also linear. From which, we can deduce that the Right Ascension of the Sun may be estimated by...

$$\alpha^{hrs} = \begin{pmatrix} 18.6974 & + 3.8198 & \times \theta \\ & + 24.00051 & \times Cycle \\ & - EoT^{\,mins}_{Gnomonical} & / \; 60 \end{pmatrix} \bmod 24 \ldots\ldots\ldots \text{Eqn 3.9}$$

This yields the Sun's Right Ascension to +/- 4 $^{\text{seconds of time}}$ from 2000 to 2050, if 4 harmonic terms are used for EoT. This reduces to +/-17 secs of time is 3 harmonics are used, and to +/1 35 seconds of time if equation 3.5 is used

Appendix

Fig 6 is a simple spreadsheet example, just looking at 25 Date/EoT pairs spread evenly every 14.6 (= 365 / 25) days over a year. The input was taken from the MICA program $^{\text{Ref 3}}$. Just the first and second harmonics were calculated and the output was generated as the sum of those two components, together the average value.



INPUT				FIRST HARMONIC				SECOND HARMONIC				OUTPUT	
				Harmonic n	1			Harmonic n	2				
Date & Time	Equation of Time mins	Step	Phase θ radians	n x θ	EoT x sin(n x θ)	EoT x cos(n x θ)	H1 = 1st Harmonic Component = A x sin{n x (θ + φ)}	n x θ	EoT x sin(n x θ)	EoT x cos(n x θ)	H2 = 2nd Harmonic Component = A x sin{n x (θ + φ)}	Fourier EoT = Av + H1 + H2	Error secs
01-Jan-2015 12:00	3.4250	0	0.0000	0.0000	0.0000	[illegible]	[illegible]	0.0000	0.0000	[illegible]	[illegible]	3.1117	19
16-Jan-2015 02:24	9.5000	1	0.2513	0.2513	2.3626	[illegible]	1.3712	0.5027	4.5767	[illegible]	[illegible]	8.9651	32
30-Jan-2015 16:48	13.2533	2	0.5027	0.5027	6.3848	[illegible]	3.1242	1.0053	11.1902	[illegible]	[illegible]	12.8512	24
14-Feb-2015 07:12	14.1500	3	0.7540	0.7540	9.6863	[illegible]	4.6808	1.5080	14.1221	[illegible]	[illegible]	14.1324	1
28-Feb-2015 21:36	12.4917	4	1.0053	1.0053	10.5471	[illegible]	5.9434	[illegible]	11.3028	[illegible]	[illegible]	12.7788	-17
15-Mar-2015 12:00	8.9833	5	1.2566	1.2566	8.5437	[illegible]	6.8325	2.5133	5.2803	[illegible]	[illegible]	9.3585	-23
30-Mar-2015 02:24	4.6717	6	1.5080	1.5080	4.6624	0.2933	7.2923	3.0159	0.5855	-4.6348	-2.4012	4.8815	-13
13-Apr-2015 16:48	0.5417	7	1.7593	1.7593	0.5321	-0.1015	7.2939	3.5186	-0.1994	-0.5036	-6.7439	0.5403	0
28-Apr-[illegible] 07:12	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	4.0212	1.8775	1.5532	-9.4183	[illegible]	[illegible]
12-Ma[illegible] 21:36	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
27-Ma[illegible] 12:00	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
11-Jun[illegible] 02:24	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	5.5292	0.3468	-0.3693	-3.7184	[illegible]	[illegible]
25-Jun-2015 16:48	[illegible]	12	3.0159	3.0159	0.3294	[illegible]	[illegible]	6.0319	-0.6536	2.5458	1.1752	[illegible]	[illegible]
10-Jul-2015 07:12	5.2967	13	3.2673	3.2673	-0.6638	-5.2549	-0.4552	6.5345	[illegible]	5.1303	5.7780	5.3131	-1
24-Jul-2015 21:36	6.5217	14	3.5186	3.5186	-2.4008	-6.0637	-2.2655	7.0372	[illegible]	4.7541	8.9514	6.6762	-9
08-Aug-2015 12:00	5.6733	15	3.7699	3.7699	-3.3347	-4.5898	-3.9335	7.5398	5.3957	1.7532	9.9103	5.9671	-18
23-Aug-2015 02:24	2.8117	16	4.0212	4.0212	-2.1664	-1.7922	-5.3544	[illegible]	2.7619	-0.5269	8.4176	3.0536	-15
06-Sep-2015 16:48	-1.6233	17	4.2726	4.2726	1.4688	0.6912	-6.4387	8.5451	-1.2508	1.0348	4.8425	-1.6059	-1
21-Sep-2015 07:12	-6.7517	18	4.5239	4.5239	6.6321	1.2651	-7.1186	9.0478	-2.4855	6.2775	0.0694	-7.0588	18
05-Oct-2015 21:36	-11.6267	19	4.7752	4.7752	11.6037	-0.7300	[illegible]	9.5504	1.4572	11.5350	[illegible]	-12.0816	27
20-Oct-2015 12:00	-15.1483	20	5.0265	5.0265	14.4069	-4.6811	[illegible]	10.0531	8.9040	12.2553	[illegible]	-15.4746	20
04-Nov-2015 02:24	-16.4417	21	5.2779	5.2779	13.8822	-8.8099	[illegible]	10.5558	14.8769	7.0005	[illegible]	-16.3562	-5
18-Nov-2015 16:48	-14.8467	22	5.5292	5.5292	10.1632	-10.8228	[illegible]	11.0584	14.8174	-0.9322	[illegible]	-14.3832	-28
03-Dec-2015 07:12	-10.4117	23	5.7805	5.7805	5.0159	[illegible]	[illegible]	11.5611	8.7909	-5.5788	[illegible]	-9.8442	-34
17-Dec-2015 21:36	-3.8483	24	6.0319	6.0319	0.9570	-3.7274	[illegible]	12.0637	1.8539	-3.3723	[illegible]	-3.6002	-15
	Av = Average of Column above				p = 2 x Average of Column above	q = 2 x Average of Column above			p = 2 x Average of Column above	q = 2 x Average of Column above			
	-0.0097				7.3362	-0.4682			9.2539	3.5897			
					Harmonic Amplitude A = √(p x p + q x q)	Harmonic Phase φ = atan2(p , q) / n			Harmonic Amplitude A = √(p x p + q x q)	Harmonic Phase φ = atan2(p , q) / n			
					7.3511	-0.0637			9.9258	0.1850			

Fig. 6. Example Spreadsheet. Follow the blue numbers. The yellow boxes show the formulae to be used in each column.

Fig 7 compares the black line, generated by 2 harmonics calculated from just those 25 points marked with the ‘x’s. The fit is close but not visually exact. Fig 8 shows the very close match achieved by using 365 input values to generate four harmonics.

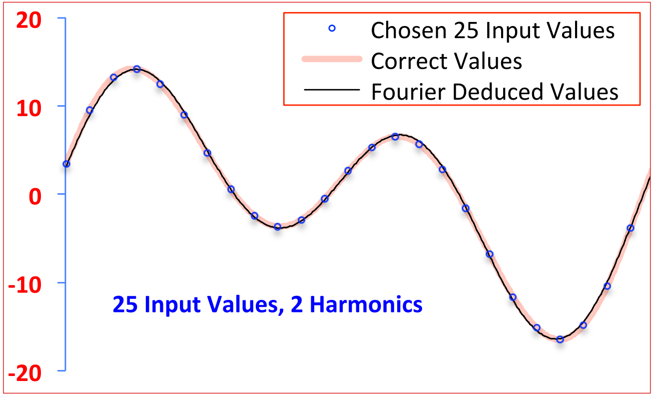


Fig. 7. The black line shows the EoT generated for 25 ‘correct’points. The pink line shows 365 ‘correct’ points. Note how close the black line follows the pink curve.

Note 1

Fourier series can be quoted as a sum of sine &/or cosine curves thus...

$$\sum_1^N a_n \times \sin(n\times\theta + b)$$

$$\sum_1^N a_n \times \cos(n\times\theta - b') \text{ where } b' = \pi/2 - b$$

$$\sum_1^N q\times\sin(n\times\theta) + r\times\cos(n\times\theta)$$

where $q = a_n \times \cos(b)$ & $r = a_n \times \sin(b)$
....................Eqn 3.10

All are trigonometrically the same

Note 2

The following Microsoft Excel function macros can be copied into a module in Excel’s Visual Basic Editor. Then, on a spreadsheet, they can be called by filling in a formula, such as

=EoT(YYYY,MM,DD,HH,MM,SS)
=Decl(YYYY,MM,DD,HH,MM,SS)
=RA(YYYY,MM,DD,HH,MM,SS)

YYYY = Year, MM = Month, etc. Note that Date and Time must be UTC.

To avoid errors in copying these out, they can be found as text files at Ref[4]

```
‘ ****************************************
‘ EoT Macro
Function  EoT(The_Year,  The_Month,  The_Day,  The_
Hour, The_Minute, The_Second)
bbb = 367 * The_Year - 730531.5
ccc = Int((7# * Int(The_Year + (The_Month + 9) /
12)) / 4)
ddd = Int(275 * The_Month / 9) + The_Day
D2000 = bbb - ccc + ddd + (The_Hour + The_Minute /
60 + The_Second / 3600) / 24
Cycle = Int(D2000 / 365.25)
```

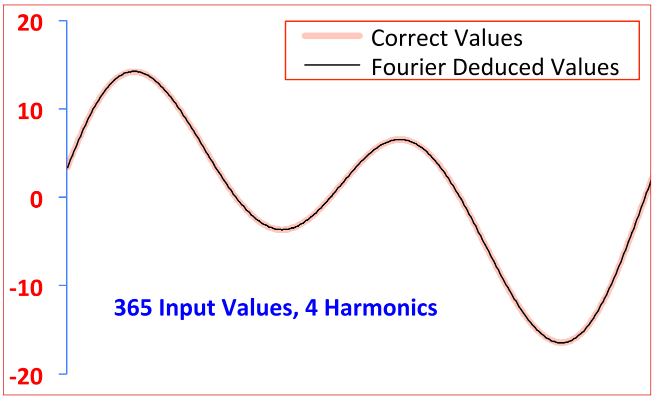


Fig. 8. As more input points and more harmonics are used, the estimation gets better and better.

```
Theta = 0.0172024 * (D2000 - 365.25 * Cycle)
Average = 0.00526
Amp1 = 7.36303 - Cycle * 9e-05
Amp2 = 9.92465 - Cycle * 0.00014
Phi1 = 3.07892 + Cycle * -0.00019
Phi2 = -1.38995 + Cycle * 0.00013

EoT1 = Amp1 * Sin(1 * (Theta + Phi1))
EoT2 = Amp2 * Sin(2 * (Theta + Phi2))
EoT3 = 0.3173 * Sin(3 * (Theta - 0.94686))
EoT4 = 0.21922 * Sin(4 * (Theta - 0.60716))

EoT = Average + EoT1 + EoT2 + EoT3 + EoT4
End Function

‘ ******************************************
‘ Declination Macro
Function  Decl(The_Year,  The_Month,  The_Day,
The_Hour, The_Minute, The_Second)
bbb = 367 * The_Year - 730531.5
ccc = Int((7# * Int(The_Year + (The_Month + 9)
/ 12)) / 4)
ddd = Int(275 * The_Month / 9) + The_Day
D2000 = bbb - ccc + ddd + (The_Hour + The_Min-
ute / 60 + The_Second / 3600) / 24
Cycle = Int(D2000 / 365.25)
Theta = 0.0172024 * (D2000 - 365.25 * Cycle)
Amp1 = 23.2639 - Cycle * 0.000131 + 0.0024 * Sin(Cycle
* 0.335103 - 0.4)
Amp1 = 23.2639 - Cycle * 0.000131 + 0.0024 * Sin(Cycle
* 0.335 - 0.4)
Phi1 = -1.38819 + Cycle * 0.000135
Decl1 = Amp1 * Sin(1 * (Theta + Phi1))
Decl2 = 0.380897 * Sin(2 * (Theta - 0.720483))
Decl3 = 0.171178 * Sin(3 * (Theta - 0.347175))
Decl4 = 0.008067 * Sin(4 * (Theta - 0.272216))
Decl = 0.37657 + Decl1 + Decl2 + Decl3 + Decl4
End Function

‘ ******************************************
‘ Right Ascension Function
Function RA(The_Year, The_Month, The_Day, The_Hour,
The_Minute, The_Second)
bbb = 367 * The_Year - 730531.5
ccc = Int((7# * Int(The_Year + (The_Month + 9) /
12)) / 4)
ddd = Int(275 * The_Month / 9) + The_Day
D2000 = bbb - ccc + ddd + (The_Hour + The_Minute /
60 + The_Second / 3600) / 24
Cycle = Int(D2000 / 365.25)
Theta = 0.0172024 * (D2000 - 365.25 * Cycle)
Average = 0.00526
Amp1 = 7.36303 - Cycle * 9e-05
Amp2 = 9.92465 - Cycle * 0.00014
Phi1 = 3.07892 + Cycle * -0.00019
Phi2 = -1.38995 + Cycle * 0.00013
```

```
EoT1 = Amp1 * Sin(1 * (Theta + Phi1))
EoT2 = Amp2 * Sin(2 * (Theta + Phi2))
EoT3 = 0.3173 * Sin(3 * (Theta - 0.94686))
EoT4 = 0.21922 * Sin(4 * (Theta - 0.60716))

EOT_hrs = (Average + EoT1 + EoT2 + EoT3 + EoT4) / 60
RA = (18.6974 + 3.8198 * Theta + 24.00051 * Cycle
- EOT_hrs)
RA = RA - (24 * (RA \ 24))
If RA < 0 Then RA = RA + 24
End Function
```

References

1. http://www.sundialsoc.org.uk/Glossary/equations/equations-new.php as of Sept 2014.
2. Jean Meeus: *Astronomical Algorithms:* Willman-Bell, Richmond (1998) - quoted on page 166.
3. *Multiyear Interactive Computer Almanac - 1800 - 2050:* US Naval Observatory: (2012). This is a high precision astronomical program, that (e.g.) provides EoT to an accuracy of 0.1 second.
4. http://www.precisedirections.co.uk/Sundials

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